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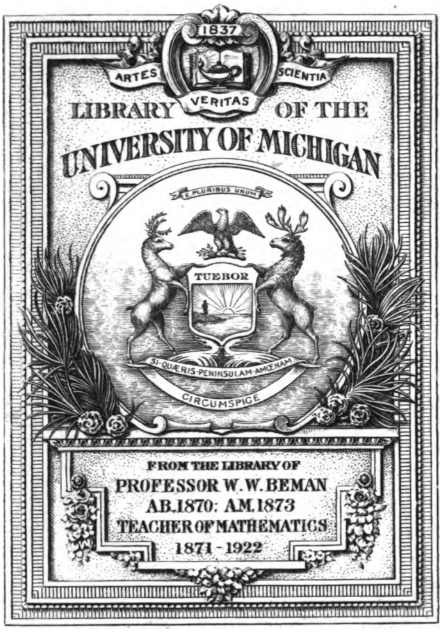

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I am happy to have here an opportunity to remark, that this method of solving equations, the discovery of which I have, in my Algebra, attributed to Mr. Holdred, appears, upon further examination, to be, in reality, due to Mr. Horner. This latter gentleman's important paper in the *Philosophical Transactions* did not fall in my way till long after I had thoroughly examined Mr. Holdred's Tract, and had myself supplied, by the aid of simple elementary means, the demonstration of the truth of the process recommended in that Tract, as it was there given without any satisfactory investigation. Upon subsequently looking over Mr. Horner's method, and perceiving that his train of reasoning was very different from my own, and that the practical rule derived from it was not exactly the same, I hastily, and, as it seems, very improperly, concluded that the two methods were essentially different, and quite independent of each other. I have since more attentively examined Mr. Horner's paper, and I find that the method given in Mr. Holdred's Tract, *without demonstration*, is identical to one satisfactorily established in the paper of Mr. Horner, but which, it appears, that gentleman passed by without particular notice, deeming his final result higher up in the scale of improvement.

ELEMENTS

OF

GEOMETRY.

LONDON:
PRINTED BY THOMAS DAVISON, WHITEFRIARS.

ELEMENTS

OF

GEOMETRY,

WITH

NOTES.

ohn adford

BY J. R. YOUNG,

AUTHOR OF AN ELEMENTARY TREATISE ON ALGEBRA.

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PREFACE.

Elements of geometry are by no means numerous in this country, a circumstance to be attributed to the almost universal preference given to Euclid; not, indeed, because the elements of Euclid is a faultless performance, but because its blemishes are so inconsiderable when compared with its extraordinary merits, that to reach higher perfection in this department of science has been generally supposed to be scarcely within the bounds of possibility, an opinion which the fruitless efforts of succeeding geometers to establish a better system have in a great measure confirmed. The superiority of Euclid's performance consists chiefly in the rigorous and satisfactory manner in which he establishes all his assertions, preferring in every case the most elaborate reasoning rather than weaken the evidence of his conclusions by the introduction of the smallest assumption.

On the continent, however, this high opinion of Euclid does not appear to prevail, and the rigour and elegance of his demonstrations seem to be less appreciated. In all the modern French treatises on geometry, it is easy to discover a wide departure from that rigorous and accurate mode of reasoning so conspicuous in the writings of the ancient geometer. From this imputation even the celebrated *Elémens de Géométrie* of *Legendre*, "the first geometer in Europe," is not exempt, notwithstanding the masterly manner in which he has treated certain difficult parts of the subject. The greatest difficulty, however, in the whole compass of geometry is doubtless the doctrine of geometrical proportion. The manner in which Euclid establishes this doctrine is remarkable for the same rigour of proof that manifests itself throughout the other parts of his work, although it is universally acknowledged that from the difficulty of the subject his reasoning is so subtle and intricate, that to beginners it opposes a very serious obstacle. The grand aim, therefore, of geometers has been to deliver this part of Euclid's performance from its peculiar difficulties, without destroying the rigour and universality of his conclusions. All attempts to accomplish this important object have been unsuccessful; and those who have abandoned Euclid's method, and have treated the subject in a more concise and easy way, have greatly fallen short of that accuracy of reasoning so essential to geometrical investigations, and have arrived at conclusions that are not indisputably established, but only approximately true: — such is the doctrine of proportion as treated by geometers of the present day. It appears, therefore, that notwithstanding the recent translation of Legendre's celebrated work into our own language—the unqualified praise which has been bestowed upon it, and its extensive circulation throughout Europe, there are still blemishes to be removed and defects to be supplied; for extraordinary as it may appear, Legendre has not, unfortunately, exercised his powerful talents upon the doctrine of proportion, but has entirely excluded the consideration of it from his elements, referring the student for requisite information "to the common treatises on arithmetic and algebra*." Now books on arithmetic and algebra can unfold the properties of proportion only as regards *numbers*, and numbers cannot extend to all classes of geometrical magnitudes, for some when compared are found to be incommensurable. The doctrine of proportion, therefore, in reference to these latter, cannot be rigorously inferred from any thing that may be established with regard to numbers or commensurable magnitudes.

Having adverted to these defects it remains for me now to give some brief account of the present attempt, and to state wherein I have endeavoured to render it more particularly worthy of examination.

And first it may be remarked, in reference to the general plan of the work, that I have taken a more enlarged and comprehensive view of the elements of geometry than I believe has hitherto been done, as I have paid particular attention to the *converse* of every proposition throughout these elements, having demonstrated the converse wherever such demonstration was possible, and in other cases shown that it necessarily failed. There can be no doubt that this comprehensive mode of proceeding, embracing as it does every thing connected with the subject, must afford the student entire satisfaction, and must also increase the accuracy, as well as the extent, of his geometrical knowledge; since he not only learns that under certain conditions a certain property must have place, but also whether or not it is possible for the same property to exist under any change of those conditions. The first, and I believe the only work in which converse propositions are fully considered, is that of *M. Garnier*, entitled *Réciproques de la Géométrie*, and which, it appears, was intended to accompany the geometry of Legendre. In the present performance I have, in several instances, availed myself of this work of Garnier, although, in many other cases, I have found it expedient to adopt a different course. The only book on geometry with which I am acquainted, where the converse accompany the direct propositions to any extent, is the *Elémens de Géométrie par M. Develey*, a very comprehensive performance; but in many instances the converse propositions are not noticed, and in but very few cases is their failure shown to take place. This plan, therefore, is not systematic and uniform.

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* Dr. Brewster's translation of Legendre, page 48.

With regard to other, and more particular improvements, introduced into this work, may be noticed proposition XIII. of the first book, taken, with little alteration, from the *Principes Mathématiques* of *M. da*

Cunha; and which, as Professor Playfair remarked, in the Edinburgh Review, vol. XX., is a decided improvement in elementary geometry, as it dispenses with an awkward subsidiary proposition of Euclid.

Upon the doctrine of proportion, which constitutes the fifth book of these elements, I have bestowed much labour and attention, and have, I hope, in some degree succeeded in diminishing the difficulties hitherto attendant upon that important subject.

The notes appended to this first part may, I think, be consulted by the student with advantage. I have therein endeavoured to point out some remarkable errors and inconsistencies into which modern geometers have fallen, particularly in reference to the theory of parallel lines, and the doctrine of proportion; and I believe many of these errors have hitherto remained unnoticed. A singular instance of this is shown in the notes to the sixth book, where a proposition in *Simpson's Geometry*, which has been for upwards of seventy years received as genuine, and adopted by more modern geometers, is proved to be false! Other instances of incautious reasoning are adduced from *Legendre*, *Dr. Simson*, and others, which it is doubtless of importance to detect and point out to the student, as indisputable proofs of the great caution necessary in geometrical reasoning.

Throughout the whole I have earnestly endeavoured to render this performance suitable to the wants of the student, and deserving of the approbation of the geometer. I can truly say that its composition has been attended with a great sacrifice, both of labour and expense; and its progress has been frequently interrupted by opposing circumstances. But if, notwithstanding, I shall have succeeded in rendering it worthy of notice, I shall consider myself fully recompensed for the pains it has cost me, and shall feel encouraged to proceed with more confidence and ardour in the remaining part of the subject.

J. R. YOUNG.

June 1*st*, 1827.

The second part will contain the GEOMETRY OF PLANES AND SOLIDS, with notes and an appendix on the SYMMETRICAL POLYEDRONS of Legendre.

⁂ The student is recommended to correct the following errors with a pen, particularly those which occur in pages 72 and 76.

ERRATA.

Page 1, bottom line, for *with*, read *to*.
15, line 2 from bottom, for bAC, read bAc.
16, line 9, for C, read B.
72, lines 6 and 7, for $R > S$, read $R > S'$.
ib. for $P > Q$ in line 6, read $P > Q'$.
ib. in lines 9 and 15, for Q', S, read Q', S'.
76, lines 10 and 12, for *oftener*, read *more or less often*.
78, line 13, for *three*, read *four*.
91, line 6 from bottom, for C, read E.
108, supply the line GE in the diagram.
109, line 18 from bottom, for $AB^2 + AC^2$, read $AB \cdot AC$.
131, line 5, for Oad, read OaD.

ELEMENTS OF GEOMETRY.

BOOK I.

GEOMETRY is the science which treats of the properties, relations, and measurement of magnitude in general. Magnitude can have but three dimensions, length, breadth, and thickness, all of which are necessary to constitute a body, or solid. It is important, however, to consider magnitude under the three distinct denominations of *lines*, *surfaces*, and *solids*, and thus the science of Geometry becomes divided into three principal branches: the first part treating of lines described upon the same plane, and of the surfaces which they enclose; the second of lines situated in different planes, and of the relations of these planes to each other; and the third part contemplating body under its several dimensions of length, breadth, and thickness. Lines are obviously the boundaries of surfaces, and surfaces are the boundaries of solids: it is equally obvious that a line, being mere length, without either breadth or thickness, can exist *only* as the boundary of a surface, and that a surface being absolutely without thickness, can exist *only* as an attribute of body. Although, therefore, it cannot be supposed that a line, or a surface, can have separate or independent existence, the fact will not in the smallest degree interrupt or embarrass our reasonings in considering these several attributes of body or space, each apart from the others, nothing more being requisite than the abstracting these others from our inquiry; so that in considering lines, length only is recognized, and in contemplating surfaces, length and breadth are combined, and thickness excluded. Having made these preliminary remarks, which were deemed essential to the student, we may proceed with the definitions.

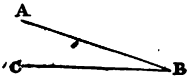
DEFINITIONS.

1. *Straight lines* are those of which but one can be drawn from one point to another.

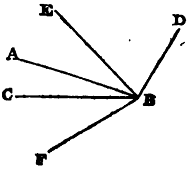
Two straight lines, therefore, cannot include space.

2. When two straight lines meet, the opening between them is called an *angle;* the point of meeting is called the *vertex,* and the lines themselves, which are said to *contain* the angle, are called the *sides* of the angle.

An angle is referred to simply, by means of the letter, at its vertex. Thus the angle contained by the straight lines AB, BC, is designated as the angle B. When, however, two or more angles have the same vertex, then, in order to denote any one in particular, it becomes necessary to specify its sides by employing the three letters at their extremities; that at the vertex being always placed in the middle.

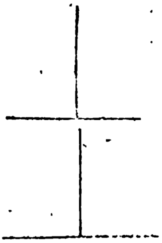


Thus the angle CBA or ABC, denotes that particular angle having the vertex B and contained by the sides AB, CB, and by the angle DBC or CBD, is in like manner meant the angle whose vertex is B, and whose sides are BD, BC.



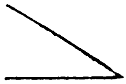
It is obvious that the quantity of an angle depends not upon the length, but entirely upon the position of its containing sides; for the opening between the sides AB, CB, must remain the same, however these lines be increased or diminished.

3. One straight line is said to be *perpendicular* to another, when it makes with it equal adjacent angles. A perpendicular at the extremity of a line, is that which makes an angle with it equal to the adjacent angle, which would be formed by prolonging the line beyond that extremity.

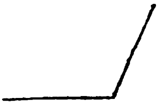


4. A *right angle* is the angle formed by a straight line and a perpendicular to it.

5. An *acute angle* is less than a right angle.



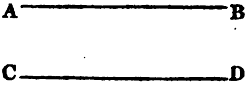
6. An *obtuse angle* is greater than a right angle.



7. A *plane surface*, or simply a *plane*, is that in which, if any two points whatever be taken, the straight line which joins them will lie wholly in it.

8. A straight line is said to be *parallel* to another when they are in the same plane, and can never meet, however far they may be produced.

If, for example, the straight line AB, how far soever it be produced, can never meet the prolongation of CD, which is in the same plane, it is said to be parallel to it.



9. By the *distance* of a point from a straight line, is meant the perpendicular from that point to the line; and one line is said to be equi-distant from another, when every point therein is equally distant from it.

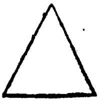
10. A *plane figure* is an enclosed plane surface.

11. If it be bounded by straight lines only, it is called a *rectilineal figure*.

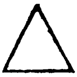
12. A *polygon* is a name used to comprehend every rectilineal figure, without regard to the number of its sides. The boundary of the figure is called its *perimeter*.

13. Among polygons, however, are more particularly distinguished the figure of three sides, called a *triangle*, and that of four sides called a *quadrilateral*.

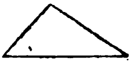
14. An *isosceles triangle* is one, which has two equal sides.



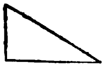
15. An *equilateral triangle* is one which has all its sides equal.



16. When no two sides are equal the triangle is called *scalene*.

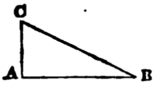


17. A *right angled triangle* is one which has a right angle.



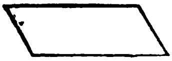
18. In a right angled triangle the side opposite the right angle is called the *hypothenuse*.

If, for example, the angle A is right, the side BC is the hypothenuse.

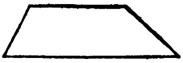


Any side of a triangle may be considered as its *base*, but it is usual, in the case of the isosceles triangle, to confine this term to that side which is not equal to either of the others.

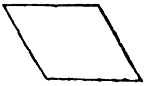
19. A *rhomboid* or *parallelogram* is a quadrilateral whose opposite sides are parallel.



20. If only two of the opposite sides are parallel, the figure is a *trapezium*.

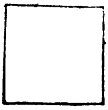


21. A *rhombus* is a rhomboid, two of whose adjacent sides are equal.



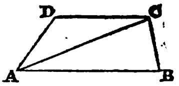
22. A *rectangle* is a rhomboid having a right angle.

23. And a *square* is a rhombus having a right angle.



24. The straight line which joins the vertices of two opposite angles of a quadrilateral, is called a *diagonal*.

Thus the line AC joining the vertices of the opposite angles DAB, DCB of the quadrilateral ABCD, is a diagonal.



25. Plane figures are *equal* when, by supposing them to be applied to each other, they would coincide throughout; and they are said to be *equivalent* when they enclose equal portions of space, and are at the same time incapable of such coincidence.

An *Axiom* is a self-evident truth.

A *Postulate* requires us to admit the possibility of an operation.

A *Theorem* is a truth, the evidence of which depends upon a train of reasoning.

The reasoning by which a truth is established, is called a *demonstration*. It is a *direct* demonstration when the truth is inferred directly from the premises as the conclusion of a regular series of inductions. The demonstration is *indirect* when the conclusion shows that the introduction of any supposition, contrary to the truth advanced, necessarily leads to an absurdity.

A *Problem* proposes an operation to be performed.

A *Lemma* is a subsidiary truth, the evidence of which must be established preparatory to the demonstration of a succeeding theorem.

A *Proposition* is a general term for either a theorem, a problem, or a lemma.

A *Corollary* is an obvious consequence, resulting from a demonstration.

An *Hypothesis* is a supposition, and may be either true or false.

A *Scholium* is a remark subjoined to a demonstration.

AXIOMS.

1. Magnitudes which are equal to the same, are equal to each other.
2. Magnitudes which are double, triple, &c., of the same, or of equal magnitudes, are equal to each other.
3. Magnitudes which are each one half, one third, &c., of the same or of equal magnitudes, are equal to each other.
4. If equals be either added to, or taken from, equals, the results will be equal.
5. But if equals be either added to, or taken from, unequals, the results will be unequal.
6. The whole is greater than a part.
7. The whole is equal to the sum of the parts into which it is divided.

POSTULATES.

1. Grant that a straight line may be drawn from one point to another.

2. And that it may be either increased till it be equal to a greater straight line, or diminished till it be equal to a less.

3. Grant also, that an angle may be increased till it be equal to a greater angle, or diminished till it be equal to a less.

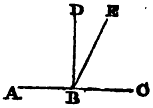
4. And lastly, that from a point either within or without a straight line, a perpendicular thereto may be drawn.

It is necessary to remark, that in the first eight books of these elements, the lines concerned in each proposition are supposed to be all situated in the same plane.

PROPOSITION I. THEOREM.

From the same point, in a given straight line, more than one perpendicular thereto cannot be drawn.

Let BD be perpendicular to the straight line AB, or AC, BC being the production of AB, and if the truth of the theorem be denied, let some other line, as BE drawn from the same point B, be also perpendicular to AC.

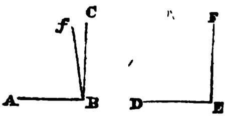


Then because the angles ABD, CBD are equal, (Def. 3.) the angle ABD must be greater than the angle EBC (Ax. 6.). But BE is perpendicular to AC, by hypothesis, therefore the angle EBC must be equal to the angle ABE (Def. 3.). It follows, therefore, that the angle ABD is greater than the angle ABE, a manifest absurdity; therefore BE cannot be perpendicular to AC.

PROPOSITION II. THEOREM.

All right angles are equal to each other.

Let ABC be a right angle, and DEF any other right angle, then, if it be denied that these two angles are equal to each other, one of them, as ABC, must be supposed greater than the other, so that DEF must be equal to some portion of ABC.

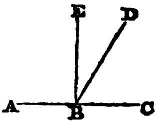


Let AB*f* represent that portion, then, because AB*f* is a right angle, B*f* is perpendicular to AB (Def. 4.); but ABC is also a right angle, therefore BC is likewise perpendicular to AB, that is from the same point B in the straight line AB, two perpendiculars thereto are drawn, which is impossible. (Prop. I.) Therefore ABC cannot be greater than DEF, and in a similar manner it may be proved that DEF cannot be greater than ABC; the two angles are, therefore, equal.

PROPOSITION III. THEOREM.

The adjacent angles which one straight line makes with another which it meets, are together equal to two right angles.

Let the straight line DB meet AC in B, making adjacent angles ABD, CBD; these angles shall together be equal to two right angles.

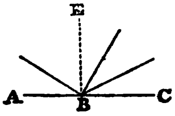


For, let BE be perpendicular to ABC, then the angle ABD is equal to the right angle ABE, together with the angle EBD, and this angle EBD, together with DBC, make up the other right angle EBC; consequently the sum of the angles ABD, CBD, is equal to two right angles.

Corollary 1. Hence, *if either side of a right angle be produced through the vertex, the adjacent angle formed will be right.*

Cor. 2. Therefore *the sides of a right angle are mutually perpendicular.*

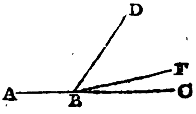
Cor. 3. *The sum of all the angles formed by straight lines drawn on the same side of another straight line from any point in it, is equal to two right angles*; for, be these angles ever so numerous, they are evidently only subdivisions of the two right angles, which a perpendicular from the point forms with the adjacent portions of the line.



PROPOSITION IV. THEOREM. (*Converse of Prop. III.*)

If, to the point where two straight lines meet, a third be drawn, making with them adjacent angles, which are together equal to two right angles; the two lines so meeting form but one continued straight line.

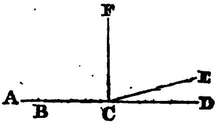
Let the two straight lines AB, CB, meet in the point B, to which let a third DB be drawn, so that the adjacent angles DBA, DBC, may together be equal to two right angles, then will ABC be one straight line.



For, if it be denied, let BF, and not BC, be the continuation of AB; then the angles ABD, FBD, are together equal to two right angles (Prop. III.). But the angles ABD, CBD, are together also equal to two right angles by hypothesis; hence the angle FBD is equal to the angle CBD, a part to the whole, which is impossible; therefore BC is the production of AB.

PROPOSITION V. THEOREM.

Two straight lines having two points common to both, form but one continued straight line.



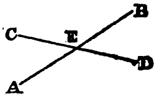
Let A, B, be the two points through which two straight lines pass, then they must necessarily coincide between A and B (Def. 1.); but if they do not coincide throughout, let ACD be the direction of one, and ACE that of the other; and at the point C, where they separate, let there be CF perpendicular to ACD.

Then, because CF is perpendicular to the straight line ACD, FCD, is a right angle (Def. 3.); and since by hypothesis ACE is a straight line, and FCA a right angle, the angle FCE is also a right angle (Prop. III. Cor. 1.), but all right angles are equal to each other (Prop. II.); therefore the whole angle FCD is equal to the part FCE, which is absurd.

Cor. Hence it follows that *if two points in a straight line be equally distant from another straight line, the former shall be equally distant from the latter throughout;* for if an equidistant straight line be drawn through these points, this line and the former will have two points common, they must, therefore, coincide.

PROPOSITION VI. THEOREM.

If two straight lines intersect each other, the opposite angles formed at their intersection will be equal.

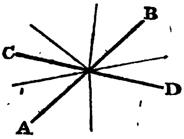


If the two straight lines AB, CD, intersect at E, the opposite angles CEB, AED, will be equal.

For the sum of the angles CEA, CEB, is equal to two right angles (Prop. III.). Also the sum of the angles CEA, AED, is equal to two right angles; that is, the sum of the angles CEA, CEB, is equal to the sum of the angles CEA, AED; and taking away from each of these equal sums the common angle CEA, the remaining angles CEB, AED, must be equal.

In a similar manner it might have been shown that the opposite angles CEA, DEB, are equal.

Cor. 1. Hence (Prop. III.) *the sum of the angles formed by the intersection of two straight lines is equal to four right angles.*



Cor. 2. And, therefore, *the amount of all the angles formed by the meeting of any number of straight lines in the same point is equal to four right angles,* since they are only so many subdivisions of the angles formed by the intersection of AB, CD.

PROPOSITION VII. THEOREM. (*Converse of Prop. VI.*)

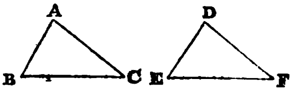
If the opposite angles formed by four straight lines meeting in a point are equal, these lines shall form but two straight lines.

Let the four straight lines AE, BE, CE, DE, meet in the point D (see the diagram to last proposition), so that the opposite angles CEB, AED, may be equal; and also the other opposite angles CEA, DEB; then AEB and CED shall be straight lines.

For, since the sum of the angles CEA, CEB, is by hypothesis equal to the sum of the angles DEB, DEA; and the sum of all four is, by the corollary to last proposition, equal to four right angles; it follows that each of the above sums must be equal to two right angles, so that the straight line CE makes, with the two AE, BE, adjacent angles, which are together equal to two right angles; therefore AEB is a straight line (Prop. IV.). In a similar manner it may evidently be proved that CED is a straight line; hence the four lines form but two distinct straight lines.

PROPOSITION VIII. THEOREM.

If two sides, and the included angle in one triangle be equal to two sides, and the included angle in another triangle, those triangles shall be equal.



Let the triangles ABC, DEF, have the sides AB, AC, and the included angle A in the one equal to the two sides DE, DF, and the included angle D in the other; then shall the angle B be equal to the angle E; the angle C equal to the angle F, and the side BC equal to the side EF.

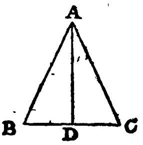
For, since AB is equal to DE, and the angle A equal to the angle D, a triangle equal to DEF may be conceived to be formed, of which AB shall be one side, and A an angle; now it is obvious that the side of this triangle, which corresponds to DF, must fall upon AC, otherwise the angles A and D would be unequal; nor can this side extend beyond or fall short of the point C, for DF is equal to AC. The two extremities, therefore, of the base would coincide with the points B, C; the two bases, therefore, would coincide throughout (Prop. V.), so that the triangle so formed would entirely coincide with the triangle ABC, and it is at the same time equal to the triangle DEF; hence the triangles ABC, DEF, are equal.

Cor. If a perpendicular from one of the angles of a triangle to the opposite side bisect that side; it shall also bisect the angle, and the sides containing that angle shall be equal, that is, the triangle will be either isosceles or equilateral.

PROPOSITION IX. THEOREM.

The angles opposite the equal sides of an isosceles triangle are equal.

Let the sides AB, AC, of the triangle ABC be equal, then will the angle C be equal to the angle B.



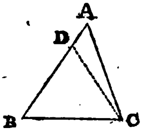
For, let AD be the line bisecting the angle A; then, in the two triangles ABD, ACD, two sides AB, AD, and the included angle in the one are equal to the two sides AC, AD, and the included angle in the other; hence the angle B is equal to the angle C (Prop. VIII.).

Cor. 1. It also follows (Prop. VIII.) that BD is equal to CD, and that the angle ADB is equal to the angle ADC; therefore *the line bisecting the vertical angle of an isosceles triangle bisects the base at right angles;* and conversely, *the line bisecting the base of an isosceles triangle at right angles, bisects also the vertical angle.*

Cor. 2. Every equilateral triangle is also equiangular.

PROPOSITION X. THEOREM. (*Converse of Prop. IX.*)

If two angles of a triangle are equal, the opposite sides are equal.

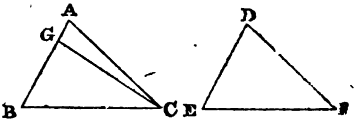


In the triangle ABC let the angles ABC, ACB, be equal; then, if it be supposed that one of the opposite sides as AB is longer than the other AC, let BD be equal to AC; then the triangle DBC is obviously less than the triangle ABC. But, since CB, BD, and the included angle, are equal to BC, CA, and the included angle, by hypothesis, it follows that the same triangles are equal (Prop. VII.), which is impossible; therefore AB cannot be longer than AC, and in a similar manner it may be shown that AC cannot be longer than AB; therefore these two sides are equal.

Cor. Therefore *every equiangular triangle is equilateral.*

PROPOSITION XI. THEOREM.

Two triangles are equal, if two angles and the interjacent side in the one, are equal to two angles and the interjacent side in the other.



Let the triangles ABC, DEF, have the two angles ABC, ACB, and the interjacent side BC, in the one, equal respectively to the two angles E, F, and interjacent side EF in the other, the two triangles will be equal.

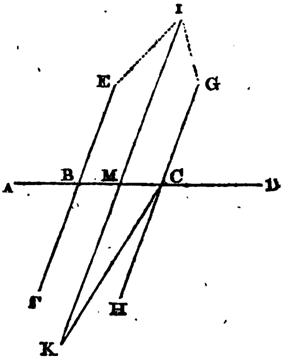
It will be necessary only to show that the side AB must be equal to the side DE (Prop. VII.). If this equality be denied, let one of these sides as AB be supposed longer than the other, and let BG be equal to ED. Join GC, then, since GB, BC, and the included angle B, are respectively equal to DE, EF, and the included angle E, the angle BCG must be equal to the angle F (Prop. VII.); but by hypothesis, the angle F is equal to the angle ACB; hence, then the angle GCB is equal to the angle ACB, a part to the whole, which is absurd; therefore, AB cannot be longer than DE, and in like manner it may be shown that DE cannot be longer than AB; AB is therefore equal to DE, and consequently the triangle ABC is equal to the triangle DEF.

Cor. From this proposition immediately follows the converse of the corollary to proposition VIII., viz · *if a perpendicular from one of the angles of a triangle to the opposite side bisect this angle, it shall also bisect the side on which it falls, so that the sides including the proposed angle must be equal.* (Prop. VIII. Cor.)

PROPOSITION XII. THEOREM.

If a straight line intersect two others, and make the alternate angles equal, the two lines shall be parallel.

Let the straight line AD intersect the two straight lines EF, GH, making the alternate angles EBC, HCB equal, then is EF parallel to GH.



For if these lines are not parallel, let them meet in some point I, and through M, the middle of BC, draw IK, making MK equal to IM, and join CK. Then the triangles IMB, KMC have the two sides IM, MB, and the included angle in the one equal respectively to the two sides KM, MC, and the included angle in the other; hence the angle IBM is equal to the angle KCM (Prop. VIII.); but by hypothesis, the angle IBM is equal to the angle HCM; therefore the angle KCM is equal to the angle HCM, so that CK, CH must coincide, that is, the line GH when produced, meets IK in two points I, K, and yet does not coincide with it, which is impossible (Prop. V.): therefore the lines EF, GH, cannot meet, they are consequently parallel.

Cor. 1. Hence also, if the angles EBC, GCD be equal, the lines EF, GH will be parallel, that is, *if a straight line intersecting two others make an exterior angle equal to the interior, opposite one on the same side of the cutting line, the two lines shall be parallel.*

Cor. 2. It follows, too, that if the angles ABE, DCH be equal, the lines EF, GH will be parallel, that is, *if the alternate exterior angles be equal, the two lines will be parallel.*

Cor. 3. Hence, likewise, *if the two exterior angles on the same side* (as ABE, DCG) *be together equal to two right angles, the two lines will be parallel.*

Cor. 4. Also, *if the two interior angles on the same side* (as EBC, GCB) *be together equal to two right angles, the two lines will be parallel.*

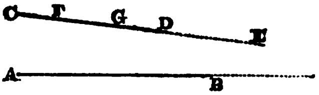
Cor. 5. Therefore *two straight lines perpendicular to a third are parallel.*

Scholium.

This last corollary shows the possibility of the existence of parallel lines (Post. 4.), and therefore also of the rhomboid.

PROPOSITION XIII. LEMMA.

If two points in a straight line be unequally distant from another straight line, the former, by being produced on the side of the least distance, shall continue to approach nearer and nearer to the latter, or its production, till at length it shall meet it.



Let the two points F, G in the straight line CD be unequally distant from the straight line AB, then the line CD by being produced, shall approach nearer and nearer to AB or its prolongation till it meets it.

For CDE cannot, at any point of its progress, discontinue to approach, and then proceed at equal distance from AB, as is manifest from the corollary to Prop. V.; still less could it, after approaching, reverse its direction, and recede from the line ABF; for it is the obvious characteristic of a straight line to preserve invariably *one* direction, however far it be extended, so that if the straight line CD be directed to a point infinitely distant, it will, by being infinitely prolonged, at length arrive at, and terminate in, that point, and must necessarily in its progress onward pass through every straight line crossing its path; and it is plain, that its continual approach toward any straight line is an indication that such line does cross its path, and must therefore be eventually intersected by it.

Cor. 1. It therefore follows that *if one straight line be parallel to another, it must be every where equidistant from it.*

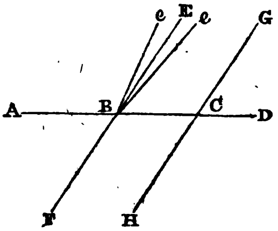
Cor. 2. Hence *from the same point more than one parallel to a straight line cannot be drawn.*

Cor. 3. And therefore *two straight lines that are parallel to the same straight line are parallel to each other*, for if they could meet, the last corollary would be contradicted.

PROPOSITION XIV. THEOREM. (*Converse of Prop. XII.*)

If a straight line intersect two parallel lines it will make the alternate angles equal.

Let AD intersect the parallels EF, GH, in B and C; the alternate angles EBC, HCB, are equal.



For if the angle EBC is not equal to the angle HCB, let the angle eBC be equal thereto, then since the alternate angles eBC, HCB are equal, Be is parallel to GH (Prop. XII.); but by hypothesis, BE is also parallel to GH, so that from the same point B, two parallels to GH are drawn, which is impossible (Prop. XIII. Cor. 2.). Hence the alternate angles EBC, HCB are equal.

Cor. 1. Therefore *a straight line intersecting two parallels makes the exterior angle equal to the interior opposite one, on the same side of the cutting line.*

Cor. 2. *The alternate exterior angles* (ABE, DCH) *are also equal.*

Cor. 3. *The two exterior angles also* (ABE, DCG) *on the same side are together equal to two right angles.*

Cor. 4. Also, *a line intersecting two parallels makes the two interior angles on the same side together equal to two right angles.*

Cor. 5. *A line perpendicular to one of two parallels is perpendicular to the other.*

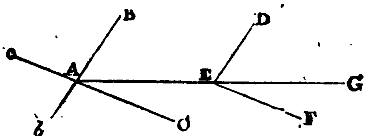
Cor. 6. Therefore, *if to each of two parallels perpendiculars be drawn, these perpendiculars shall be parallel;* for by last corollary they are all perpendicular to the same line.

Scholium.

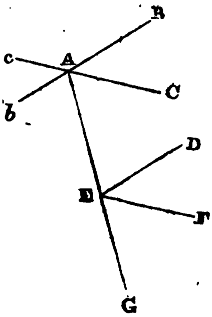
These corollaries prove the converse of the corollaries to proposition XII.

PROPOSITION XV. THEOREM.

Two angles are equal if the sides of the one are parallel, each to each, to the sides of the other, and at the same time lie each upon that side of the line joining their vertices which the parallel side lies on, or else each upon the opposite side of that line.



Let the sides of the angles BAC, DEF, be parallel each to each, and let AB lie upon the same side of AE as the parallel side ED, and let AC also lie upon the same side of this line as EF, the two angles are equal.

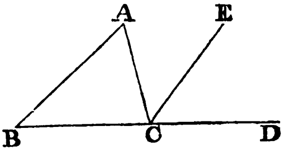


For since BA, DE are parallel, the angles BAE, DEG are equal (Prop. XIV. Cor. 1.). For similar reasons the angles CAE, FEG are also equal; therefore the angle BAC is equal to the angle DEF. If the parallel sides of the two angles lie upon contrary sides of AE, as Ab, ED, Ac, EF, it is obvious the angles will still be equal, for the angle bAC is equal to the angle BAC.

PROPOSITION XVI. THEOREM.

If any side of a triangle be produced, the exterior angle will be equal to both the interior opposite angles.

Let one of the sides as BC of the triangle ABC be produced, the exterior angle ACD is equal to both the interior opposite angles A and B.



For let CE be parallel to BA, then is the angle A equal to the alternate angle ACE (Prop. XIV.), and the angle C is equal to the exterior angle ECD (Prop. XIV. Cor. 1.); therefore the sum of the angles A and B is equal to the sum of ACE, ECD, that is, to the whole exterior angle ACD.

Cor. 1. Since the angle ACD together with ACB make two right angles, it follows that *in every triangle the sum of the three angles is equal to two right angles.*

Cor. 2. Hence *if two angles in one triangle be equal to two in another, the third angle in the one will be equal to the third angle in the other.*

Cor. 3. Therefore, and from proposition XI. *if two angles and any side in one triangle be equal to two angles and a corresponding side in another, the triangles will be equal.*

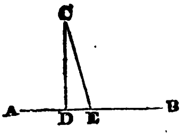
Cor. 4. *A triangle cannot have more than one angle so great as a right angle.*

Cor. 5. And therefore *every triangle must have at least two acute angles.*

Cor. 6. *In a right angled triangle the right angle is equal to the sum of the other two.*

Cor. 7. *If one of the equal sides of an isosceles triangle be produced beyond the vertex, the exterior angle will be double of either angle at the base, and if the base be produced, the exterior angle will exceed the interior adjacent angle.*

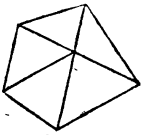
Cor. 8. *From a point without a straight line, only one perpendicular to that line can be drawn,* for if CD and CE were both perpendicular to AB, the exterior angle CEB would be equal to the single interior opposite angle CDE.



PROPOSITION XVII. THEOREM.

In any polygon the sum of all the angles is equal to twice as many right angles as the figure has sides, all but four right angles.

For if from the vertices of the several angles, lines be drawn to any point within the figure, the polygon will obviously be divided into as many triangles as it has sides. Now, by last proposition, the sum of the angles in each triangle amounts to two right angles, therefore the angles of all the triangles are together equal to twice as many right angles as the figure has sides, that is to say, the sum of the angles of the polygon, together with those about the point within it, are equal to twice as many right angles as the polygon has sides; but those angles which are about the point, amount to four right angles, (Prop. VI. Cor. 2.) deducting these therefore, and there remains the angles of the polygon equal to twice as many right angles as the figure has sides, all but four right angles.



Cor. 1. *The angles of a quadrilateral are together equal to four right angles*, and this is the only figure, the sum of whose angles amount to as many right angles as there are sides to the figure.

Cor. 2. *If all the angles of a quadrilateral be equal, each will be a right angle.*

Cor. 3. *If the sum of two angles of a quadrilateral be equal to two right angles, the sum of the remaining two will likewise be equal to two right angles; or if the sum of two angles be equal to the sum of the remaining two, each sum will amount to two right angles.*

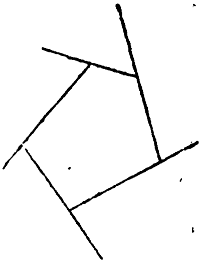
Cor. 4. *In equiangular figures of more than four sides, each angle is greater than a right angle.*

PROPOSITION XVIII. THEOREM.

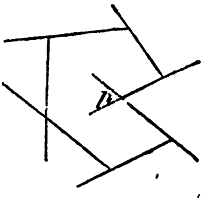
In any polygon the exterior angles, formed by producing each side, amount to four right angles.

Let each side of the polygon in the margin be produced, the exterior angles shall amount to four right angles.

For each exterior angle, together with the adjacent interior angle, make two right angles; so that the sum of all the angles, both interior and exterior, amount to twice as many right angles as there are sides to the figure, and the interior angles alone amount to this sum, all but four right angles (Prop. XVII.); therefore the exterior angles must amount to four right angles.



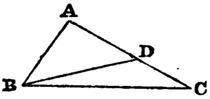
Scholium.



It is possible for a polygon to have what are called *re-entrant* angles, such as the angle *p* in the annexed figure, which, if considered as an inward angle, must exceed two right angles. The consideration, however, of angles, as greater than two right angles, does not enter into elementary geometry, and is, indeed, at variance with the definition of an angle; for an angle attains its utmost limit when the opening between its sides cannot be further increased, which will be the case when the two sides arrive at that position where they become but one straight line; so that two right angles form the ultimate extent of angular magnitude. By drawing a line from the vertex of a re-entrant angle to a point within the polygon, this angle may be divided into two others, each less than two right angles; and it is plain that proposition XVII. is applicable to polygons with *re-entrant* angles, as well as to those having only *salient* angles. The above proposition also applies equally to the former kind of polygons; but it must be observed that the sides of the re-entrant angles must not be produced through their vertices; but, in the opposite directions, for otherwise exterior angles will not be formed, but interior ones as shown in the above diagram.

PROPOSITION XIX. THEOREM.

In any triangle the greater angle is opposite to the longer side.



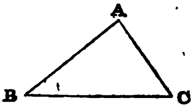
Let the side AC of the triangle ABC be longer than the side AB, and let AD be equal to AB:—join BD. Then, because the sides AB, AD, of the triangle ABD are equal, the angles ADB, ABD, are also equal (Prop. IX.); but the angle ADB, being an exterior angle of the triangle BCD, is greater than the angle C (Prop. XVI.); therefore the angle ABC, which exceeds the angle ABD, must necessarily exceed the smaller angle C; hence the greater angle is opposite to the longer side.

Cor. It follows, therefore, that *the less angle is opposite to the shorter side.*

PROPOSITION XX. THEOREM. (*Converse of Prop. IX.*)

In any triangle the longer side is opposite to the greater angle.

In the triangle ABC let the angle C be greater than the angle B, then will the side AB be longer than the side AC.



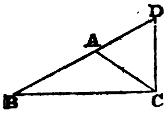
For, if AB were equal to AC, the angle C would be equal to the angle B. If AB were shorter than AC, then would the angle C be less than the angle B (Prop. XIX. Cor.). As, therefore, AB can be neither equal to, nor shorter than AC, it must necessarily be longer. The longer side, therefore, is opposite to the greater angle.

Cor. 1. Therefore *the shorter side is opposite to the less angle.*

Cor. 2. *In the right angled triangle the hypothenuse is the longest side* (Prop. XVI. Cor. 4.).

PROPOSITION XXI. THEOREM.

Any two sides of a triangle are together longer than the third side.



The two sides AB, AC, for instance, of the triangle ABC are together longer than the third side BC. For, let BA be produced till AD be equal to AC, and let DC be joined.

Then the angle ACD being equal to the angle D (Prop. IX.), the angle BCD must be greater than the angle D; consequently the side BD is longer than the side BC (Prop. XX.). But BD is equal to BA and AC together; therefore the two sides AB, AC, are together longer than BC.

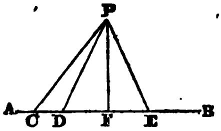
Cor. Therefore AC is longer than the difference between AB, BC; and AB longer than the difference between AC, BC; that is, *any side of a triangle exceeds the difference between the other two.*

PROPOSITION XXII. THEOREM.

The perpendicular drawn from a point to a straight line is shorter than any other line drawn thereto from the same point; and those lines which meet

the proposed line at equal distances from the perpendicular are themselves equal, and the more remote from the perpendicular the point of meeting is, the longer is the line drawn.

Let PF be perpendicular to the straight line AB, and let PD, PE, intercept equal distances DF, EF; let also any other oblique line PC be drawn.



First, since the angle PFE is right, PE is longer than PF (Prop. XX. Cor. 2.); therefore the perpendicular is shorter than any oblique line.

Again, since DF, FP, and the included angle are equal to EF, FP, and the included angle PD is equal to PE (Prop. VIII.); hence lines drawn from P intercepting equal distances from the perpendicular are equal.

Lastly, because the triangle PDE is isosceles, as we have just shown, the exterior angle PDC is greater than the inward adjacent angle PDE (Prop. XVI. Cor. 7.), and this last is greater than the angle PCD (Prop. XVI.); therefore PDC being greater than PCD, PC is longer than PD (Prop. XX.); that is, the more remote from the perpendicular any oblique line falls, the longer it is.

Cor. 1. It follows from this last case that *two equal straight lines cannot be drawn from a point to a line, and fall upon the same side of the perpendicular from that point to the line;* and that, therefore, *it is impossible to draw three equal straight lines from the same point to a given straight line.*

Scholium.

The converse of this proposition immediately follows, that is,

First, *The shortest line that can be drawn from a point to a straight line is a perpendicular thereto;* for by the first part of the preceding demonstration, if this were not the perpendicular, it could not be the shortest line.

Secondly, *If equal lines be drawn from a point to a line, the distances intercepted between them and the perpendicular from that point will be equal;* for, by the third case of the above, if the distances intercepted were unequal, the lines drawn would be also unequal.

And, lastly, *If unequal lines be drawn, the longer shall fall more remotely from the perpendicular;* for, if it were less remote it would be shorter, if equally remote, equal, as we have already proved.

Cor. 2. Hence, *if from a point to a line two lines be drawn, then, that which is not shorter than the other shall exceed any line drawn between them.*

Cor. 3. And *the shorter of two lines so drawn shall be shorter than any line drawn between them.*

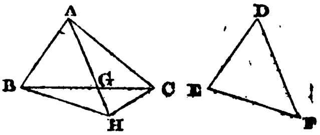
Cor. 4. *A line drawn from the middle of another to a point equally distant from its extremities, is a perpendicular thereto;* for a perpendicular, from the point to the line, is equally distant from its extremities (Schol.).

Cor. 5. Therefore, *if through two points, of which each is equally distant from the extremities of a straight line, a second line be drawn, it shall be perpendicular to the first;* for, by last corollary, a line from the middle of the former to either point is a perpendicular.

Cor. 6. It, moreover, follows that *two right angled triangles are equal, when the hypothenuse and one side in the one triangle are respectively equal to the hypothenuse and a side in the other.*

PROPOSITION XXIII. THEOREM.

If two sides of one triangle be respectively equal to two sides of another, but include a greater angle; the third side of the former shall exceed the third side of the latter.



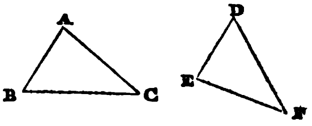
Let ABC, DEF, be two triangles having any two sides, as AB, AC, in the one, respectively equal to two sides DE, DF, in the other, while the angle included by the former is greater than that included by the latter; then will the third side BC, of the former, be longer than the third side EF, of the latter.

Of the two sides AB, AC, let AC be that which is not shorter than the other, let the line AGH make an angle with AB equal to the angle D, let AH be equal to DF or AC, and let BH, HC, be drawn.

Since AC is not shorter than AB, it is longer than AG (Prop. XXII. Cor. 2.); therefore, as AH is equal to AC, the extremity H must fall below the line BC. The angles ACH, AHC, are equal (Prop. IX.); hence the angle BCH is less than the angle AHC, and, therefore, necessarily less than BHC; hence the side BC is longer than the side BH (Prop. XX.); but BH is equal to EF, because the sides AB, AH, and the included angle are respectively equal to DE, DF, and the included angle (Prop. VIII.); consequently BC is longer than EF.

PROPOSITION XXIV. THEOREM. (*Converse of Prop. XXIII.*)

If two sides of one triangle be respectively equal to two sides of another, while the third side of the former is longer than that of the latter, the angle included by the former two sides shall exceed that included by the latter two.



In the triangles ABC, DEF, let the two sides AB, AC, be equal respectively to the two sides DE, DF; while the side BC exceeds the side EF, the angle A will exceed the angle D.

For the angle A cannot be equal to the angle D, for then the side BC would be equal to the side EF (Prop. VIII.); nor can it be less, for then BC would be less than EF (Prop. XXIII.). As, therefore, the angle A can be neither equal to, nor less than, the angle D, it must necessarily be greater.

PROPOSITION XXV. THEOREM.

Two triangles are equal which have the three sides of the one respectively equal to the three sides of the other.

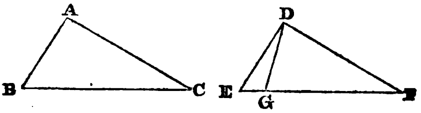
For the angle included between any two sides in the one triangle must be equal to the angle included by the two corresponding sides in the other; since, if it were unequal, the opposite sides would be unequal (Prop. XXIII.), which is contrary to the hypothesis; therefore the three angles in the one triangle are respectively equal to the three angles in the other.

Cor. From this, and Prop. VIII, it follows that *one quadrilateral is equal to another, if the sides of the one are respectively equal to the sides of the other: and the angle included by any two sides of the one also equal to the angle contained by the two corresponding sides of the other.*

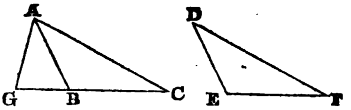
PROPOSITION XXVI. THEOREM.

Two triangles are equal, if two sides, and an opposite angle in one are respectively equal to two sides; and a corresponding opposite angle in the other, provided the other opposite angles in each triangle are either both acute, or both obtuse.

In the triangles ABC, DEF, let the sides AB, AC, be respectively equal to the sides DE, DF, and let the angle C, opposite to the side AB, be equal to the angle F, opposite to the side DE; then will BC be equal to EF, provided the angles B and E are either both acute, or both obtuse.



First, let B and E be acute, then, if the equality of BC and EF be denied, one of them as EF must be longer than the other. Let, then, FG be taken equal to BC, and draw DG, which will be equal to AB (Prop. VIII.), and, therefore, equal to DE; consequently the angle E is equal to the angle DGE (Prop. IX.); DGE is, therefore, an acute angle, but this angle, together with DGF, make up two right angles (Prop. III.); DGF is, therefore, an obtuse angle. But, since the triangles ABC, DGF, are equal, the angle DGF must be equal to the angle B, and, therefore, acute, which is impossible; so that FG cannot be equal to BC, and the demonstration would have been the same had BC been supposed longer than EF; these two sides are, therefore, equal.

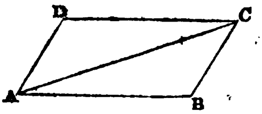


Next, let the angles B and E be obtuse; then, if EF be supposed longer than BC, produce the latter beyond the vertex B, till CG be equal to EF:—join AG. Then, as before shown, AG is equal to DE, or to AB, and the angle AGC, which is equal to the angle ABG (Prop. IX.), is acute, since ABC is obtuse; but the same angle must be obtuse, because the triangles AGC, DEF, are equal (Prop. VIII.) which is impossible; whence EF cannot be longer than BC, and had BC been supposed longer than EF a similar absurdity would obviously have followed; hence in this case also the sides BC, EF, are equal, and, therefore, (Prop. XXV.) the triangle ABC is equal to the triangle DEF.

PROPOSITION XXVII. THEOREM.

The opposite sides and angles of a rhomboid are equal.

Let ABCD be a rhomboid, the opposite sides and angles are equal.



Draw the diagonal AC, then, since AB, DC are parallel, the alternate angles BAC, DCA, are equal (Prop. XIV.), and because AD, BC, are also parallel, the alternate angles

DAC, BCA, are likewise equal; hence the two angles BAC, DAC, are together equal to the two angles DCA, BCA; that is, the opposite angles BAD, DCB, are equal. Again, since the angles BAC, BCA, and the interjacent side of the triangle ABC are respectively equal to the angles DCA, DAC, and the interjacent side of the triangle CDA, the triangles are equal (Prop. XI.); therefore the side AB is equal to the side CD, the side BC to DA, and the angle B to the angle D; hence, in a rhomboid, the opposite sides and angles are equal.

Cor. 1. From this proposition, and Cor. 3. to Prop. XVII., it follows, that *if one angle of a rhomboid be right, all the angles will be right.*

Cor. 2. Therefore *in the rectangle and square* (see Definitions) *all the angles are right, and in the latter all the sides are equal.*

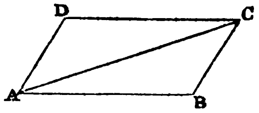
Cor. 3. *The diagonal divides a rhomboid into two equivalent triangles.*

Cor. 4. *Parallels included between two other parallels are equal.*

PROPOSITION XXVIII. THEOREM. (*Converse of Prop. XXVII.*)

If the opposite sides of a quadrilateral be equal, or if the opposite angles be equal, the figure will be a rhomboid.

In the quadrilateral ABCD let the opposite sides be equal, the figure will be a rhomboid.



Let the diagonal AC be drawn, then the triangles ABC, ADC, are equal, since the three sides of the one are respectively equal to those of the other (Prop. XXV.); therefore the angles BAC, DCA, opposite the equal sides BC, DA, are equal; therefore DC is parallel to AB; the angles ACB, CAD, opposite the equal sides BC, DA, are also equal; BC is, therefore, parallel to AD (Prop. XII.); hence ABCD is a rhomboid.

Next, let the opposite angles be equal.

Then the sum of the angles BAD, ADC, must be equal to the sum of the angles DCB, CBA; therefore each sum is equal to two right angles (Prop. XVII Cor. 3.); therefore AB, DC, are parallel (Prop. XII. Cor. 3.). For similar reasons AB, BC, are parallel; therefore the figure is a rhomboid.

PROPOSITION XXIX. THEOREM.

If two of the opposite sides of a quadrilateral are both equal and parallel, the figure is a rhomboid.

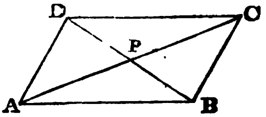
In the quadrilateral ABCD (preceding diagram), let AB be equal and parallel to DC, then will ABCD be a rhomboid.

For the diagonal AC makes the alternate angles BAC, DCA, equal (Prop. XIV.); so that in the triangles ABC, CDA, two sides, and the included angle in each are respectively equal; these triangles are, therefore, equal (Prop. VIII.), the angle ACB is, therefore, equal to the angle CAD; hence AD is parallel to BC (Prop. XII.), and the other two sides are parallel by hypothesis; therefore ABCD is a rhomboid.

Cor. If, in addition, the parallel sides be each equal to a third side, the rhomboid will be either a rhombus or a square, according as it has, or has not, a right angle.

Scholium.

It has been proved (Prop. VIII.) that two triangles are equal when two sides, and the included angle in the one are respectively equal to two sides, and the included angle in the other; we may now infer further, that *two triangles are equivalent, or equal in surface, when two sides of the one are respectively equal to two sides of the other, and the sum of the included angles equal to two right angles.*



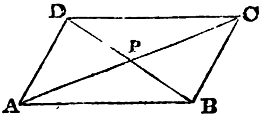
For, let the triangles ADC, BCD, having two sides AD, DC, in the one equal to the two BC, CD, in the other be placed as in the margin, a side of the one coinciding with the equal side in the other; let also the included angles ADC, BCD, be together equal to two right angles, and let AB, BD, be drawn.

Then, since the angles ADC, BCD, are together equal to two right angles, the lines AD, BC, are parallel (Prop. XII. Cor. 3.), but they are also equal by hypothesis; hence, by the above proposition, the figure ABCD is a rhomboid; now, the triangle ADC is half the rhomboid (Prop. XXVII. Cor. 3.), so also is the triangle BCD; these triangles are, therefore, equivalent.

PROPOSITION XXX. THEOREM.

The diagonals of a rhomboid bisect each other.

The diagonals AC, BD, of the rhomboid ABCD are mutually bisected in the point P.



For, since AB, CD, are parallel, the angles PAB, PBA, are respectively equal to the angles PCD, PDC (Prop. XIV.), and AB being also equal to CD, the triangles PAB, PCD, are equal (Prop. XI.); therefore the sides AP, CP, opposite the equal angles ABP, CDP, are equal, as also the sides BP, DP, opposite the other equal angles. The diagonals of a rhomboid, therefore, bisect each other.

PROPOSITION XXXI. THEOREM. (*Converse of Prop. XXX.*)

If the diagonals of a quadrilateral bisect each other, the figure is a rhomboid.

If the diagonals AC, BD (preceding diagram), bisect each other, ABCD is a rhomboid.

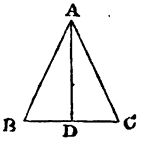
For the two sides AP, PB, and included angle being equal to the two sides CP, PD, and included angle, the side AB is equal to the side CD (Prop. VIII.). For similar reasons AD is equal to CB; hence (Prop. XXVIII.) the quadrilateral is a rhomboid.

BOOK II.

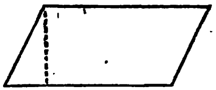
DEFINITIONS.

1. The *altitude* of a triangle is the distance of one of its sides, taken as a base, from the vertex of the opposite angle.

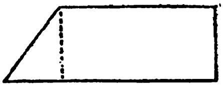
The perpendicular AD from the vertex A to the base BC, is the altitude of the triangle ABC.



2. The altitude of a rhomboid is the distance of one of its sides, considered as a base, from the opposite side.

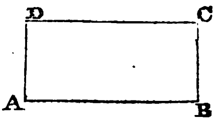


3. The altitude of a trapezium is the distance between its parallel sides.



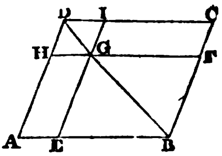
4. A rectangle is said to be *contained* by its adjacent sides.

The rectangle ABCD is contained by the sides DA, AB. For brevity it is often referred to as the rectangle of DA, AB.



5. If, within a rhomboid, two straight lines parallel to the adjacent sides be drawn so as to intersect the diagonal in the same point; then, of the four rhomboids into which the figure is divided, those two through which the diagonal passes are said to be *about the diagonal*, and the other two are called their *complements*.

Thus, of the four rhomboids into which the lines HF, IE, divide, the rhomboid ABCD, HGID, and EBFG are about the diagonal, and AEGH, GFCI, are the complements.



In referring to a rhomboid it will be sufficient to employ the letters placed at two opposite corners.

PROPOSITION I. THEOREM.

The complements of the rhomboids about the diagonal of a rhomboid are equivalent.

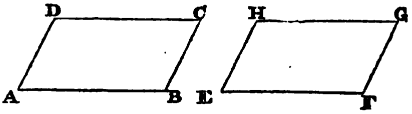
Thus, in the above diagram, the rhomboids AG, GC, are equivalent.

The triangle ABD is equal to the triangle CDB; the triangle HGD to the triangle IDG, and the triangle EBG to the triangle FGB (Prop. XXVII. Cor. 3.); take the triangles HGD, EBG, from the triangle ABD, and there will remain the rhomboid AG; take, in like manner, from the other half of the rhomboid AC the triangles IDG, FGB, equal to the former two, and there will remain the rhomboid GC; these rhomboids, therefore, are equivalent.

PROPOSITION II. THEOREM.

Rhomboids are equal which have two sides, and the included angle in each equal.

Let the sides AB, AD, and the angle A in the rhomboid AC be respectively equal to the sides EF, EH, and the angle E in the rhomboid EG; these rhomboids are equal.



For the opposite sides of rhomboids being equal, it follows that the four sides of the rhomboid AC are respectively equal to those of the rhomboid EG; therefore, since the angles A and C are also equal, the two rhomboids are equal (Prop. XXV. Cor. B. I.)

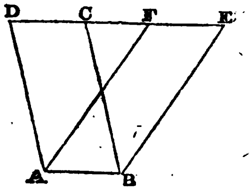
Cor. 1. *If a rhomboid and a triangle have two sides, and the included angle in the one respectively equal to two sides and* *the included angle in the other, the rhomboid will be double the triangle* (Prop. XXVII. Cor. 3. B. I.).

Cor. 2. *Rectangles contained by equal lines are equal.*

PROPOSITION III. THEOREM.

Rhomboids which have the same base and equal altitudes are equivalent.

Let the rhomboids AC, AE, standing upon the same base AB, have equal altitudes; or, which amounts to the same thing, let the opposite sides DC, FE, lie in the same line DE parallel to the base (Prop. XIII. Cor. 1. B. I.); these rhomboids are equal.



For DC is equal to FE, each being equal to AB (Prop. XXVII. B. I.); consequently DF is equal to CE: and since DA, AF, are respectively equal to CB, BE, the triangle ADF is equal to the triangle BCE. Take the former triangle from the quadrilateral ABED, and there will remain the rhomboid AE; take the latter triangle from the same space, and there will remain the rhomboid AC; these rhomboids are, therefore, equivalent.

Cor. 1. *Rhomboids whose bases and altitudes are respectively equal are equivalent,* for the equal bases being placed the one upon the other must coincide.

Cor. 2. *Triangles whose bases and altitudes are respectively equal are equivalent,* as they are the halves of equivalent rhomboids (Prop. XXVII. Cor. 3. B. I.).

Cor. 3. *Every rhomboid is equivalent to a rectangle of equal base and altitude.*

Cor. 4. *A line bisecting the opposite sides of a rhomboid divides the rhomboid into two equal parts; and a line from the middle of any side of a triangle to the vertex of the opposite angle divides the triangle into two equal parts* (Cor. 1 and 2.).

Cor. 5. Therefore *a triangle is equivalent to a rhomboid of equal base and of half its altitude, or to one of equal altitude and of half its base.*

Scholium.

1. It is very evident that the converse of the above proposi-

tion is not true, that is to say, it cannot be inferred that two equivalent rhomboids shall have their bases and altitudes equal; for it has been shown (Prop. 1.) that the rhomboids AG, GC, are equivalent (see the diagram), where the base GF must be longer than the base GE, provided BA is longer than AD, for then the angle ADB being greater than ABD (Prop. XIX. B. I.), the angle EGB, which is equal to ADB, is greater than EBD; consequently EB is longer than EG, but EB is equal to GF, therefore GF is longer than GE.

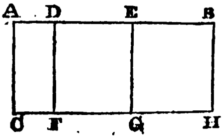
2. It is however, true, that *equivalent rhomboids upon the same base have equal altitudes*, for if the altitude of one be supposed less than that of the other, and the side opposite its base be prolonged, a portion of the other rhomboid must be cut off thereby, and the remaining portion still be equal to the former rhomboid, by the proposition, which is absurd; the altitudes therefore are equal. Having shown this, we may further prove that *equivalent rhomboids of equal altitudes have also equal bases*, for they are equivalent to rectangles of the same bases and altitudes: now any side of a rectangle may be considered as the base; taking then those sides as bases which are equal to the altitude of the rhomboids, the other sides or altitudes are, as shown above, equal, and these altitudes are the bases of the rhomboids: the bases are therefore equal.

Cor. 6. Hence, *equivalent triangles whose bases are equal, have equal altitudes; and equivalent triangles whose altitudes are equal, have equal bases* (Cor. 5.).

PROPOSITION IV. THEOREM.

If there be two straight lines of which one is divided into parts, the sum of the rectangles contained by the undivided line, and the several parts of the other, will be equal to the rectangle contained by the two whole lines.

Let the lines be AB, AC, of which the former is divided into the parts AD, DE, EB, then the rectangles contained by AC, and each of these parts, are together equal to the whole rectangle AH, contained by AB, AC.



Let DF, EG, be parallel to AC, then the angles FDE, GEB, being each equal to the angle A, the rhomboids AF, DG, EH, are rectangles, and DF, EG, being each equal to AC (Prop. XXVII. Cor.

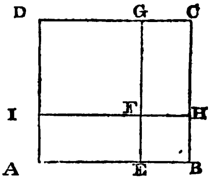
4. B. 1.), these rectangles are contained by AC, and the several parts of AB, and as they make up the whole rectangle AH, they are together equal to it.

Cor. The square of a line is equivalent to twice the rectangle of the whole line and the half thereof.

PROPOSITION V. THEOREM.

If a straight line be divided into two parts, the square described upon the whole line shall be equivalent to the squares on the two parts, together with double the rectangle contained by those parts.

The square ABCD upon the line AB, is equivalent to the squares upon any two parts AE, EB, into which the line is divided, together with double the rectangle contained by them.



Let EG be parallel to BC, BH equal to BE, and HFI parallel to BA. Then the opposite sides of the figure EH being parallel, and the angle B being right, EH is a square. Again, because AC is a square, the lines IH, EG, parallel to its sides are equal (Prop. XXVII. Cor. 2 and 4. B. I.); if then from each, the equals FH, FE, be respectively taken, the remainders IF, FG, will be equal, and F being a right angle, FD is a square; hence the containing sides of the rectangle AF are equal to those of the rectangle FC, consequently the square AC includes the squares on DG, (or AE) EB, together with double the rectangle contained by AE, EB.

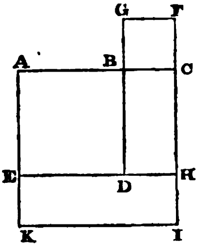
Cor. The square of a line is equivalent to four times the square of half the line.

PROPOSITION VI. THEOREM.

The square described on the difference of two lines is equivalent to the squares on the two lines diminished by twice their rectangle.

The square upon AB, the difference of the two lines AC, BC, is equivalent to the squares on these lines, diminished by twice their rectangle.

Let AD be the square on AB, BF the square on BC, and AI the square on AC, and let ED be produced to H.

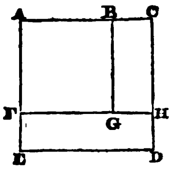


The adjacent angles GBC, CBD being right angles, BD is the continuation of GB, (Prop. IV. B. I.); CH for a similar reason is the continuation of FC, and the figure GH is a rectangle. Again, since DH is equal to FG, or GB, EH is equal to DG; also EK is equal to BC or GF, each being the excess of a side of the square AI above a side of the square AD; hence the rectangle EI is equal to the rectangle GH (Prop. II. Cor. 2.), consequently the square on AB is equivalent to the squares on AC, BC, diminished by twice the rectangle of AC, BC.

PROPOSITION VII. THEOREM.

The difference of the squares of any two lines is equivalent to the rectangle contained by the sum and difference of those lines.

The difference of the squares of the two lines AC, AB, is equivalent to the rectangle contained by AC, BC, their sum and difference.



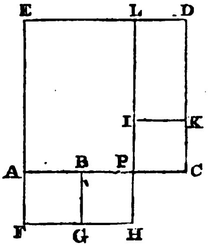
Let AD be the square on AC, and AG the square on AB, and produce FG to H, then BH is the rectangle of AB, BC; also, since FE is equal to BC, each being the excess of a side of the square AD, above a side of the square AG, the rectangle FD is contained by lines equal to AC, BC; and the two rectangles BH, FD are therefore together equal to the rectangle contained by the sum AB, AC of the two lines, and their difference BC (Prop. IV. B. II.) and these rectangles make up the excess of AD above AG.

PROPOSITION VIII. THEOREM.

The sum of the squares on two lines is equivalent to half the square on their sum, together with half the square on their difference.

The squares on the two lines AB, AC, are together equivalent to half the square on their sum, and half the square on BC their difference.

Let AG be the square on AB, and AD the square on AC, and through P, the middle of BC, let LH, parallel to EA, be drawn, meeting FG produced in H; make PI equal to PH, and draw IK parallel to PC.



Since the angles at A are right, EAF is a straight line, so that EH is a rectangle, and it is contained by lines equal to the sum of AC, AB, and half that sum AP; it is therefore equal to half the square of the sum AC, AB (Prop. IV. Cor.). Again, IP, PC, being equal, by construction, to PH, PB, the rectangles PK, PG, are equal; hence the two squares AD, AG, are together equivalent to the two rectangles EH, LK; now, CK being equal to BG or BA, KD is equal to BC, for EC is a square, the rectangle LK is thus contained by lines equivalent to BC, and the half thereof PC, and is consequently equal to half the square on BC; it therefore follows that the squares AD, AG, are together equivalent to half the square on the sum of AC, AB, and half the square on BC, the difference of AC, AB.

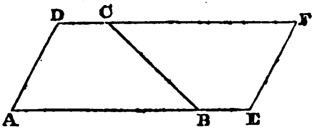
Cor. Hence, *twice the sum of the squares of two lines is equivalent to the squares of their sum and difference.*

PROPOSITION IX. THEOREM.

A trapezium is equivalent to a rectangle contained by its altitude, and half the sum of its parallel sides.

The trapezium ABCD is equivalent to the rectangle contained by its altitude, and the sum of its parallel sides AB, DC.

Produce DC and AB till CF be equal to AB, and BE to DC, and join EF, completing the rhomboid AF (Prop. XXIX. B. I.).



The four sides of the quadrilateral AC are respectively equal to those of the quadrilateral FB, and at the same time the angle A is equal to the angle F (Prop. XXVII. B. I.); therefore these quadrilaterals are equal (Prop. XXV. Cor. B. I.). Hence the trapezium AC is equivalent to half the rhomboid

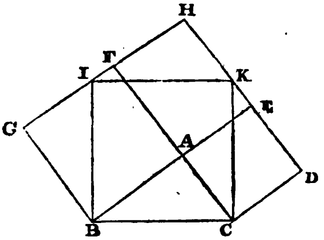
AF, or to a rectangle of the same altitude, and half the base AE (Prop. III. Cor. 3. B. II.), that is, the trapezium is equivalent to a rectangle of the same altitude, and whose base is half the sum of the parallel sides.

Cor. If two points equally distant from the opposite corners of a rhomboid be taken on the opposite sides, the line which joins them divides the rhomboid into two equal parts.

PROPOSITION X. THEOREM.

The squares described on the two sides of a right angled triangle, are together equivalent to the square described on the hypothenuse.

Let the triangle ABC be right angled at A, then the sum of the squares AG, AD, described on the sides, will be equal to the square on the hypothenuse BC.



Produce GF, DE, till they meet at H; let BI, CK, be each perpendicular to BC, and join IK. Because CBI, ABG, are both right angles, if ABI be taken from each, there will remain the angle CBA equal to the angle IBG; and AG being a square, AB is equal to BG; therefore the triangle BGI is equal to the triangle BAC (Prop. XI. B. I.). Exactly in a similar manner may it be shown that the triangle CDK is also equal to the triangle BAC; hence BI, BC, CK, are all equal; therefore BK is a square, IBC being a right angle. Again, since the sides HI, IK, are respectively parallel to the sides AB, BC, and lie on the same side of IB, the angle HIK is equal to the angle ABC (Prop. XV. B. I.); for similar reasons the angle HKI is equal to the angle ACB, the side IK is also equal to BC; hence the triangle HIK is equal to the triangle ABC (Prop. XI. B. I.). It has, therefore, now been proved that the three triangles BGI, IHK, KDC, are equal to each other, and to the triangle ABC; it follows, therefore, that the square on BC is equivalent to the whole space BGHDC, diminished by three times the triangle ABC. Now, the rectangle AH contained by the sides AE, AF, equal to AC, AB, is double the triangle ABC (Prop. II. Cor. 1. B. II.); hence the squares AG, AD, are together likewise equal to the whole space BGHDC, diminished by three times the triangle ABC; consequently these two squares are equivalent to the square on BC.

Cor. 1. Hence *the square on either side of a right angled triangle is equivalent to the square on the hypothenuse, diminished by the square on the other side.*

Cor. 2. Hence, also, *the square of a side is equivalent to the rectangle contained by the sum and difference of the hypothenuse and other side* (Prop. VII.).

Cor. 3. *The square on the diagonal of a square is double the square on a side.*

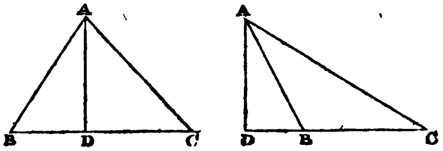
Cor. 4. *The squares of the sides of a rectangle are together equivalent to the squares of the diagonals.*

Cor. 5. If a triangle be divided into two right angled triangles by means of a perpendicular from the vertex to the base, then, since the square of each hypothenuse is equivalent to the square of the adjacent portion of the base, together with the square of the perpendicular, it follows, that *in any triangle the difference of the squares of the sides is equivalent to the difference of the squares of the parts into which the perpendicular from the vertex divides the base.*

PROPOSITION XI. THEOREM.

In any triangle, the square of a side opposite an acute angle is less than the squares of the base and of the other side, by twice the rectangle contained by the base and the part of it included between the perpendicular and the vertex of the acute angle.

In the two triangles ABC, let BC in each be considered as base; then, whether the perpendicular AD fall within or without the triangle, the square of AB opposite the acute angle C shall in either case be equivalent to the squares of AC, BC, diminished by twice the rectangle of BC, CD.



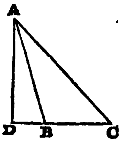
The square of AB is equivalent to the squares of BD, DA (Prop. X.); now, BD is the difference of the two lines BC, DC; the square of BD is, therefore, equivalent to the squares of BC, DC, diminished by twice the rectangle of BC, DC (Prop. VI.). Hence the square of AB is equivalent to the squares of the three lines

AD, BC, DC, diminished by twice the rectangle of BC, DC, that is (Prop. X.), to the squares of AC, BC, diminished by twice the rectangle of BC, DC.

PROPOSITION XII. THEOREM.

In any triangle having an obtuse angle, the square of the side opposite thereto exceeds the squares of the base and other side, by twice the rectangle of the base and the distance of the perpendicular from the vertex of the obtuse angle.

In the triangle ABC, let B be an obtuse angle, AD the perpendicular on the prolongation of the base BC, then will the square of AC be equivalent to the squares of AB, BC, together with twice the rectangle of CB, BD.



For the square of AC is equivalent to the squares of CD, DA (Prop. X.), and the square of CD is equivalent to the squares of CB, BD, together with twice the rectangle of CB, BD (Prop. V.); therefore the square of AC is equivalent to the squares of the three lines CB, BD, DA, and twice the rectangle of CB, BD, that is, to the squares of CB, BA, and twice the rectangle of CB, BD.

Cor. 1. From the two last propositions the converse of proposition X. immediately follows, that is, *if the square of any side of a triangle be equivalent to the sum of the squares of the other two sides, the angle opposite the former shall be right;* for these propositions show, that if such equivalence exist, the angle can neither be acute nor obtuse.

Cor. 2 We may, moreover, readily infer the converse of these two propositions themselves, that is, first, *if in the triangle ABC* (see the diagrams to Prop. XI.) *the square of AB is equivalent to the squares of AC, BC, diminished by twice the rectangle of BC, CD, the angle C shall be acute;* for by the above proposition and proposition X., if this angle were either obtuse or right, the said equivalence could not exist. Again, *if in the triangle ABC the square of AC is equivalent to the squares of AB, BC, together with twice the rectangle of CB, BD, the angle B opposite AC shall be obtuse;* for by last proposition, and proposition X., this angle can neither be acute nor right.

Scholium.

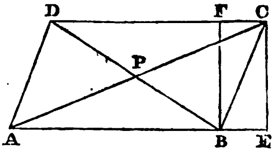
The last corollary may obviously be expressed in a more unrestricted form, thus: *If the square of any side of a triangle is less than the sum of the squares of the other two sides, the angle opposite the former side is acute, but if it is greater than that sum, the opposite angle is obtuse.*

PROPOSITION XIII. THEOREM.

The squares of the sides of a rhomboid are together equivalent to the squares of the diagonals.

In the rhomboid ABCD, the squares of AB, BC, CD, DA, are together equivalent to the squares of AC, BD.

The truth of this for the rectangle has already been established (Prop. X. Cor. 4.). Let then the angles ABC, ADC, be obtuse, and consequently the other angles acute.



Let BF be perpendicular to DC, and CE perpendicular to the production of AB.

Then, by last proposition, the square of AC is equivalent to the squares AB, BC, together with twice the rectangle of AB, BE; and (by Prop. XI.) the square of DB is equivalent to the squares of BC, DC, diminished by twice the equal rectangle DC, FC: for DC, FC, are respectively equal to AB, BE.

Hence, adding the squares of AC, BD, together, the sum is equivalent to the squares of AB, BC, CD, DA.

Cor. 1. Half the sum of the squares, that is, the squares of AB, BC, or of DC, CB, is equivalent to half the squares of BD, CA, that is, to twice the squares of BP, CP, (Prop. V. Cor.); hence, *in any triangle, whether having an obtuse angle, as ABC, or having all its angles acute, as DBC, the sum of the squares of the two sides is equivalent to twice the squares of half the base, and of the line from the vertex to the middle of the base.*

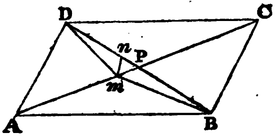
Cor. 2. Hence also, *in any triangle, the squares on the sum and difference of the sides are equivalent to the squares of the base, and of twice the line from the vertex to the middle of the base.* (Prop. VIII. Cor.).

PROPOSITION XIV. THEOREM. *(Converse of Prop. XIII.)*

If the squares of the sides of a quadrilateral be together equivalent to the squares of the diagonals, the figure shall be a rhomboid.

In the quadrilateral ABCD, let the squares of the sides be equivalent to the squares of the diagonals, the figure is a rhomboid.

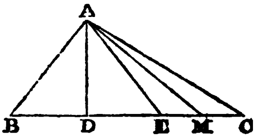
If it be not a rhomboid, the diagonals AC, BD, cannot bisect each other (Prop. XXXI. B. I.), let then *m* be the middle of AC, and *n* the middle of BD; join D*m*, *m*B, and *mn*.



Then, by Cor. 1. last proposition, the squares of AD, DC, are together equivalent to twice the squares of A*m*, D*m*; and the squares of AB, BC, are together equivalent to twice the squares of C*m*, B*m*; it therefore follows, that the squares of the four sides are equivalent to the squares of AC, and twice the squares of D*m*, B*m*. But, by hypothesis, the squares of the sides are equivalent to the squares of AC, BD; hence this latter square must be equal to twice the squares of D*m*, B*m*: but these are equivalent to the square of DB, together with four times the square of *mn* (Prop. XIII. Cor. I.); hence *mn* can have no value, that is, the middle of each diagonal must be one common point: therefore the figure is a rhomboid. (Prop. XXXI. B. I.)

Scholium.

The converse of the corollaries to proposition XIII. do not obtain. It will be sufficient to show this, with respect to the first corollary, the converse of which is as follows: If the sum of the squares of two sides of a triangle be equivalent to twice the square of a line, from the vertex to the base, together with twice the square of one of the parts, into which it divides the base; the base shall be divided in the middle.



Let AD be the perpendicular from the vertex to the base of the triangle ABC, let DE be equal to BD, join AE, and let M be the middle of EC; then if AM be drawn, the squares of AC, AE, will be equivalent to twice the squares of AM, CM, but the square of AE is equal to the square of AB, since DE is equal to BD; therefore the squares of AC, AB, are equivalent to the squares of AM, CM, although M is not the middle of the base BC.

BOOK III.

DEFINITIONS.

1. Every line which is not straight is called a *curve* line.

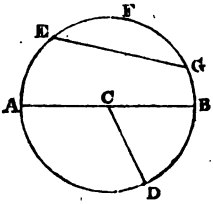
2. A *circle* is a space enclosed by a curve line, every point in which is equally distant from a point within the figure; which point is called the *centre.*

3. The boundary of a circle is called its *circumference.*

4. A *radius* is a line drawn from the centre to the circumference.

5. A *diameter* is a line which passes through the centre, and has its extremities in the circumference.

A diameter, therefore, is double the radius.



In the circle AEFBD, of which C is the centre, CD is the radius, and AB the diameter.

6. An *arc* is any portion of the circumference.

7. The *chord* of an arc is the straight line joining its extremities. It is said to *subtend* the arc.

8. A *segment* of a circle is the portion included by an arc and its chord.

The space EFGE included by the arc EFG, and the chord EG is a segment; so also is the space included by the same chord and the arc EADBG.

9. A *sector* of a circle is the portion included by two radii and the intercepted arc.

The space CBDC is a sector of the circle.

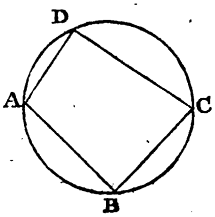
10. A *tangent* is a line which touches the circumference, that is, it has but one point in common with it, which point is called the *point of contact.*

11. One circle *touches* another when their circumferences have one point in common, and only one.

12. A line is *inscribed* in a circle when its extremities are in the circumference.

13. An angle is *inscribed* in a circle when its sides are inscribed.

14. A polygon is *inscribed* in a circle when its sides are inscribed, and under the same circumstances the circle is said to *circumscribe* the polygon.



Thus AB is an inscribed line, ABC an inscribed angle, and the figure ABCD is an inscribed quadrilateral.

15. A circle is *inscribed* in a polygon when its circumference touches each side, and the polygon is said to be *circumscribed about the circle.*

16. By *an angle in a segment* of a circle is to be understood, an angle whose vertex is in the arc, and whose sides intercept the chord; and by *an angle at the centre* is meant one whose vertex is at the centre. In both cases the angles are said to be *subtended* by the chords or arcs which their sides include.

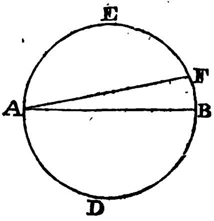
POSTULATE.

From any point as a centre with any radius, a circumference may be described.

PROPOSITION I. THEOREM.

A diameter divides a circle and its circumference into two equal parts; and, *conversely*, the line which divides the circle into two equal parts is a diameter.

Let AB be a diameter of the circle AEBD, then the portions AEB, ADB, are equal both in surface and boundary.



Suppose the portion AEB were to be applied to the portion ADB, while the line AB still remains common to both, there must be an entire coincidence; for if any part of the boundary AEB were to fall either within or without the boundary ADB, lines from the centre to the circumference could not all be equal. Therefore a diameter divides the circle and its circumference in two equal parts.

Conversely, the line dividing the circle into two equal parts is a diameter.

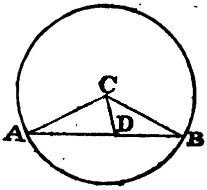
For, let AB divide the circle into two equal parts, then, if the centre is not in AB, let AF be drawn through it, which is, therefore, a diameter, and, consequently, divides the circle into two equal parts; hence the portion AEF is equal to the portion AEFB, which is absurd.

Cor. The arc of a circle, whose chord is a diameter, is a semi-circumference, and the included segment is a semi-circle.

PROPOSITION II. THEOREM.

Any line inscribed in a circle lies wholly within the circle.

Let the line AB have its extremities in the circumference of a circle, whose centre is C; this line shall lie wholly within the circle.



For, to whatever point D, between the extremities of AB, a line CD from the centre be drawn, it must be shorter than CA or CB (Prop. XXII. Cor. 2. B. I.); AB, therefore, lies wholly within the circle.

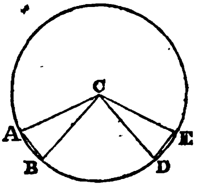
Cor. Every point, moreover, in the production of AB is farther from the centre than the circumference.

PROPOSITION III. THEOREM.

In the same, or in equal circles, equal angles at the centre are subtended by equal arcs.

Let C be the centre of a circle, and let the angle ACB be equal to the angle ECD, then the arcs AB, ED, subtending these angles are equal.

Join AB, ED.



Then the triangles ACB, DCE, having two sides and the included angle in the one, equal to two sides and the included angle in the other, are equal; so that if ACB be applied to DCE, there shall be an entire coincidence, the point A coinciding with D, and B with E; the two extremities, therefore, of the arc AB thus coinciding with those of the arc DE; all the intermediate parts must coincide, inasmuch as they are all equally distant from the centre.

Cor. 1. It follows, moreover, that *equal angles at the centre are subtended by equal chords.*

Cor. 2. *If the angle at the centre of a circle be bisected, both the arc and the chord which it subtends shall also be bisected.*

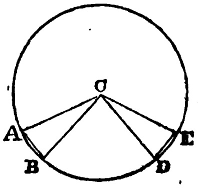
Scholium.

The above reasoning obviously applies to the case of equal circles, as the one would entirely coincide with the other.

PROPOSITION IV. THEOREM. (*Converse of Prop. III.*)

In the same circle equal arcs subtend equal angles at the centre.

For, let the arc AB be equal to the arc DE, then is the angle ACB equal to the angle DCE.



For, if the arc AB were to be applied to the arc DE, they would coincide; so that the extremities AB of the chord AB would coincide with those of the chord DE; these chords are, therefore, equal: hence the angle ACB is equal to the angle DCE (Prop. XXV. B. I.).

Cor. 1. *Equal chords subtend equal angles at the centre.*

Cor. 2. Therefore *equal chords subtend equal arcs; and,* conversely, *equal arcs are subtended by equal chords.*

Cor. 3. *The angle at the centre, subtended by half a semicircumference, is a right angle;* for the adjacent angles subtended by the two halves are equal.

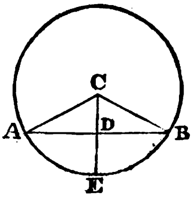
Scholium.

Similar reasoning evidently applies to equal circles.

PROPOSITION V. THEOREM.

A perpendicular from the centre of a circle to any chord bisects it, and also the arc which it subtends.

The perpendicular CDE, from the centre C to the chord AB, bisects AB.



For, if CA, CB be drawn, these lines will be equal; therefore their extremities A, B are equally distant from the perpendicular (Prop. XXII. Schol. B. I.).

It, moreover, follows (Pro. IX. Cor. 1. B. 1.) that the angle ACB is bisected by CDE; therefore, also, the arc AEB is bisected (Prop. III. Cor. 2.).

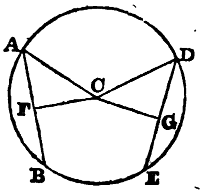
Cor. 1. Since the perpendicular, from the centre, joins the centre with the middle of the chord, or with the middle of the arc, it follows, conversely, that *the line joining the centre, and middle of the chord, or the middle of the arc, must be perpendicular to the chord.*

Cor. 2. And *a perpendicular, through the middle of the chord, passes through the centre, and through the middle of the arc, bisecting the angle which it subtends at the centre.*

PROPOSITION VI. THEOREM.

Equal chords are equidistant from the centre of the circle, and, *conversely*, equidistant chords are equal.

In the circle ABED, let the chords AB, DE be equal, then the perpendiculars CF, CG, from the centre, shall likewise be equal.



For, since the chords are bisected in F and G, (Prop. V.) AF is equal to DG; therefore the right angled triangles AFC, DGC, having the hypothenuse and a side in each equal, are equal (Prop. XXII. Cor. 6. B. I.); therefore CF is equal to CG.

Conversely, if the distances CF, CG are equal, then, in the right angled triangles AFE, DGE, there will be the hypothenuse AC, and a side CF in the one, equal to the hypothenuse DC and a side CG in the other; therefore AF is equal to DG: consequently AB, the double of AF, is equal to DE, the double of DG.

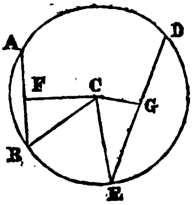
PROPOSITION VII. THEOREM.

The longer the chord is, the nearer it is to the centre; and, *conversely*, the nearer any chord is to the centre, the longer it is.

Of the two chords AB, DE, let DE be the longer, then shall DE be nearer to the centre than AB; that is, the perpendicular CG shall exceed the perpendicular CF.

For, let CB, CE be joined.

Then the triangles BFC, EGC, being right angled, and having equal hypothenuses CB, CE, the squares of CF, FB,



are together equivalent to the squares of CG, GE, (Prop. X. B. II.). But the square of GE, the half of DE, (Prop. V. B. III.) exceeds the square of FB, the half of AB, by hypothesis; therefore the square of CF must as much exceed the square of CG: otherwise the above equality could not exist. Hence, the line CF exceeds the line CG, that is, the longer chord is nearer to the centre.

Conversely, If the chord DE be nearer to the centre than AB, DE shall be longer than AB.

For the squares of BF, FC being equivalent to the squares of EG, GC, and the square of GC being, by hypothesis, less than the square of FC, it follows that the square of BF must be as much less than the square of EG; hence DE, the double of EG, is longer than AB, the double of BF.

Cor. 1. Hence *the diameter is the longest chord that can be drawn in a circle.*

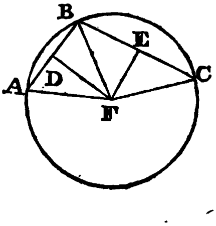
Cor. 2. *The shorter the chord is, the farther it is from the centre, and,* conversely, *the farther the chord is from the centre, the shorter it is.*

PROPOSITION VIII. THEOREM.

Through three given points, which are not in the same straight line, one circumference of a circle may be made to pass, and but one.

Let ABC, be three points not in the same straight line: they shall all lie in the circumference of the same circle.

For let the distances AB, BC be bisected by the perpendiculars DF, EF, which must meet in some point F; for if they were parallel, the lines DB, CB, perpendicular to them, would also be parallel(Prop. XIV. Cor. 6. B. I.), or else form but one straight line: but they meet in B, and ABC is not a straight line by hypothesis.



Let then FA, FB, and FC be drawn; then, because FA, FB meet AB at equal distances from the perpendicular, they are equal. For similar reasons FB, FC, are equal; hence the points A, B, C, are all equally distant from the point F, and consequently all lie in the circumference of the circle, whose centre is F, and radius FA.

It is obvious, that, besides this, no other circumference can pass through the same points, for the centre, lying in the perpendicular DF bisecting the chord AB, and at the same time in the perpendicular EF bisecting the chord BC (Prop. V. Cor. 2.), must be at the intersection of these perpendiculars; so that as there is but one centre and one radius, there can be but one circumference.

Cor. 1. As two circumferences cannot have three points in common, it follows that *one circumference cannot cut another in more points than two.*

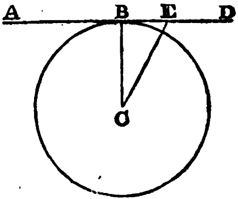
Cor. 2. Therefore *from any point not the centre more than two equal lines cannot be drawn to the circumference.*

Cor. 3. Consequently *that point from which three equal lines may be drawn is the centre.*

PROPOSITION IX. THEOREM.

A perpendicular at the extremity of a diameter is a tangent to the circle, and, *conversely*, a tangent to a circle is perpendicular to the diameter drawn from the point of contact.

Let ABD be perpendicular to the radius CB, it shall touch the circle in the point B.



For to show that this point B is the only one which the circle has in common with AD, let CE be drawn to any other point E in that line, then CE being longer than CB, (Prop. XXII. B. I.) the point E must necessarily lie without the circumference; B, therefore, is the only point in AD which is also in the circumference; AD is therefore a tangent to the circle.

Conversely. Let now AD touch the circle in B, and let BC be drawn from the point of contact to the centre; AD shall be perpendicular to BC.

For every point in AD, except B, lies without the circumference: to suppose that any point lay within the circumference, the line AD must be supposed to pass through the circle, and thus cut the circumference in some other point besides B; but this is contrary to the hypothesis; therefore a line drawn from C to any other point, E in AD, must be longer than CB; this therefore being the shortest line, is perpendicular to AD (Prop. XXII. Schol.).

Cor. 1. *From the same point in a circumference only one* *tangent can be drawn*, for two lines could not be both perpendicular to the diameter at the same point.

Cor. 2. *Tangents at each extremity of a diameter are parallel.* (Prop. XII. Cor. 5. B. I.)

Cor. 3. And conversely, *parallel tangents are both perpendicular to the same diameter* (Prop. XIV. Cor. 5. B. I.), *and have their points of contact at its extremities.*

Scholium.

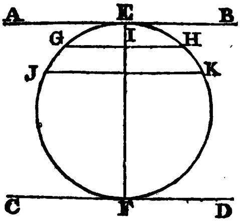
This proposition shows the possibility of the existence of a tangent to a circle (Post. IV. B. I.).

PROPOSITION X. THEOREM.

The arcs of a circle intercepted by two parallels are equal.

If the parallels are tangents to the circle, as AB, CD, then each of the arcs intercepted is a semi-circumference, as their points of contact coincide with the extremities of the diameter EF (Prop. IX. Cor. 3.).

But suppose only one, as AB, is a tangent, and that the other, GH, is a chord, which being perpendicular to FE, the arc GEH is bisected by FE, (Prop. V.) so that in this case also the intercepted arcs GE, EH are equal.



Next, let both the parallel lines be chords, as GH, JK, and let the diameter FE be perpendicular to one of them, as GH: it will also be perpendicular to the other, since they are parallel; therefore this diameter must bisect each of the arcs which they subtend, that is, GE is equal to EH, and JE to EK; therefore JE diminished by GE, is equal to EK diminished by EH; that is, in this case likewise, the intercepted arcs JG, KH, are equal.

PROPOSITION XI. THEOREM. (*Converse of Prop. X.*)

If two straight lines intercept equal arcs of a circle, and do not cut each other within the circle, the lines will be parallel.

If the two lines be AB, CD (preceding diagram), which touch the circumference, and if, at the same time, the intercepted arcs EJF, EKF are equal, EF must be a diameter

(Prop. I.); and therefore AB, CD, are parallel (Prop. IX. Cor. 2.)

But if only one of the lines, as AB, touch, while the other, GH, cuts the circumference, making the arcs EG, EH equal, then the diameter FE, which bisects the arc GEH, is perpendicular to its chord GH (Prop. V. Cor. 1.): it is also perpendicular to the tangent AB (Prop. IX.); therefore AB, GH are parallel.

If both lines cut the circle, as GH, JK, and intercept equal arcs GJ, HK, let the diameter FE bisect one of the chords, as GH: it will also bisect the arc GEH (Prop. V.), so that EG is equal to EH; and since GJ is by hypothesis equal to HK, the whole arc EJ is equal to the whole arc EK; therefore the chord JK is bisected by the diameter FE: as therefore both chords are bisected by this diameter, they are perpendicular to it, (Prop. V. Cor. 1.) that is, they are parallel.

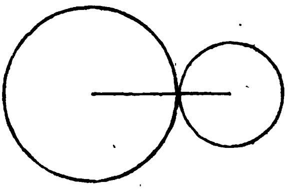
Scholium.

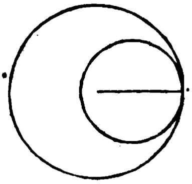
The restriction in the enunciation of this proposition, viz. that the lines *do not cut each other within the circle*, is necessary; for lines drawn through the points G, K and J, H will intercept the equal arcs GJ, HK, and yet not be parallel, since they will intersect each other within the circle.

PROPOSITION XII. THEOREM.

If two circumferences have a point in common, situated in the straight line joining their centres, the circumferences will touch each other in that point.

For if they have also another point in common, then a line joining the two points will lie wholly within each circle (Prop. II.), and therefore the two lines from the middle of this chord to the centre of each circle must both be perpendicular to it (Prop. V. Cor. 1), and must therefore form but one straight line; but by hypothesis, the centres are in the line passing through the other point, so that two distinct lines have the same extremities, which is impossible.





Cor. 1. *If two circumferences cut each other, the points of* *intersection must both be without the line joining the centres;* for if either were in this line, the circumferences would touch.

Cor. 2. *If the distance between the centres of two circles be equal to the sum of their radii, the circumferences touch externally.*

Cor. 3. *But if the distance be equal to the difference of the radii, one touches the other internally,* for in both cases the circumferences pass through the same point in the line joining the centres.

Scholium.

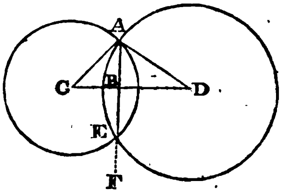
Hence the possibility of one circle touching another (Post. B. III.).

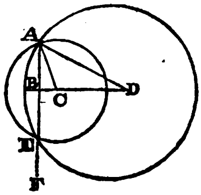
PROPOSITION XIII. THEOREM.

If two circumferences have a point in common, situated out of the line joining their centres, the circumferences will cut each other.

Let C and D be the centres of two circles, and let A be a point situated out of the line CD, common to the two circumferences.

Let AB be a perpendicular to CD, then AB, if produced, shall again cut the circumferences: for if it were a tangent to either, it would be perpendicular either to DA, or CA, which it is not; let it then cut the circumference, whose centre is C, in the point E, then the chord AE is bisected in B (Prop. V.); let it cut the other circle in F, then AF is also bisected in B; consequently the points E and F coincide, that is, the circumferences again have a point in common.





Cor. 1. Hence the converse of proposition XII. viz. *if two circumferences touch each other, their centres and point of contact lie in the same straight line;* for if the point of contact lay out of the line joining the centres, the circles would cut.

Cor. 2. Therefore, *if two circumferences touch each other, the distance of their centres is equal either to the sum or difference of their radii, accordingly as they touch externally or* *internally*, which is the converse of Corollaries 2 and 3, last proposition.

Cor. 3. It moreover follows from the above demonstration, that *the line joining the intersections of the circumferences, is bisected at right angles by the line joining the centres;* for it is shown that the perpendicular to this line, from one of the points, passes through the other, and is bisected by the line joining the centres.

Scholium.

1. Corollary 1, to proposition XII., proves the converse of this proposition.

2. A mere inspection of the preceding diagrams shows, that *if two circumferences cut, the distance of the centres must be less than the sum of the radii*; for CD is less than the sum of CA, DA (Prop. XXI. B. I.); and, consequently, that, *if the distance of the centres of two circles be greater than the sum of their radii, the circumferences will neither touch* (Prop. XIII. Cor. 2.) *nor cut.*

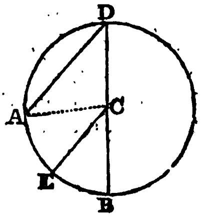
3. It is, moreover, equally plain that, *if two circumferences cut, the distance of the centres must exceed the difference of the radii;* for CD is longer than the difference of CA, DA (Prop. XXI. Cor. B. I.): consequently, *if the distance of the centres of two circles be less than the difference of their radii, their circumferences will neither touch* (Cor. 2.) *nor cut.*

4. It appears, therefore, that *in order that two circumferences may cut, the distance of their centres must be less than the sum, and greater than the difference of the radii.*

PROPOSITION XIV. THEOREM.

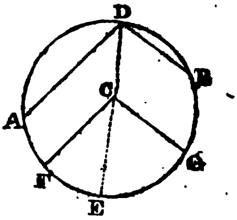
An inscribed angle is equal to an angle at the centre of the circle whose sides include half the arc included by those of the inscribed angle.

Let ADB be an inscribed angle, the sides of which include the arc AB; this angle is equal to one at the centre C, whose sides include half that arc.



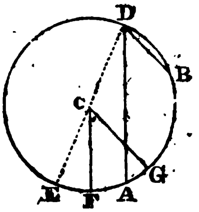
Suppose, first, that the centre lies in one of the sides DB of the proposed angle, let CA be drawn, as also CE, bisecting the angle ACB.

Then, since ACD is an isosceles triangle, the exterior angle ACB is double the interior opposite angle D; therefore the half thereof ACE is equal to D, and its sides intercept half the arc AB (Prop. III. Cor. 2.).



Next, let the centre C lie within the sides of the angle ADB, and let DCE be drawn, as also CF to the middle of the arc AE, and CG to the middle of the arc BE.

Then, by the preceding case, the angle ADE is equal to the angle FCE, and the angle EDB to the angle ECG; hence, the whole angle ADB is equal to the whole angle FCG; and the arc FG, subtended by the angle at the centre, is, by construction, equal to half the arc AEB, subtended by that at the circumference.



Lastly, let the centre lie without the sides of the angle ADB, and draw, as before, DE through the centre C; CF to the middle of the arc AE, and CG to the middle of the arc EB.

Then, by the first case, the angle ADE is equal to the angle FCE, and EDB to ECG; hence the angle ADB, the difference of the angles at the circumference is equal to FCG, the difference of the angles at the centre. Now, the arc FG is, by construction, equal to the difference of half the arcs EB, EA, or, which is the same thing, to half the difference of those arcs, that is, to half the arc AB; so that in every case the inscribed angle is equal to an angle at the centre, whose sides intercept half the arc included by those of the former.

Cor. 1. Hence *an angle at the centre of a circle is double an angle at the circumference, subtending the same arc.*

Cor. 2. *Angles in the same, or in equal segments, or, in other words, inscribed angles, subtended by the same, or equal arcs, are equal; each being equal to the angle at the centre, whose sides include half the equal arcs.*

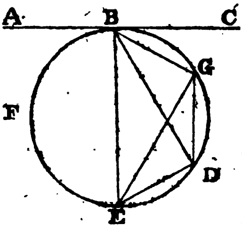
Cor. 3. *An angle in a semi-circle is a right angle;* for it is equal to an angle at the centre, subtended by half a semi-circumference (Prop. IV. Cor. 3.).

Cor. 4. *If, in a circle, two chords drawn from a point in the circumference be respectively equal to two chords drawn from another point, they shall include equal angles;* for the equal chords subtending equal arcs (Prop. IV. Cor. 2.), each angle must include the same portion of the circumference.

PROPOSITION XV. THEOREM.

The angle formed by a tangent, and a chord drawn from the point of contact, is equal to an angle in the alternate segment of the circle; that is, to an inscribed angle subtended by the arc intercepted by the sides of the former.

Let ABC touch the circle BDEF in B, making with the chord BD the angles ABD, CBD; the former will be equal to an angle in the segment BGD, and the latter to an angle in the segment BFED.



For, draw the diameter BE and the chord ED; draw also EG, BG, and DG to any point G in the arc BGD; then, ABE being a right angle (Prop. IX.), it is equal to the angle BGE in the semicircle (Prop. XIV. Cor. 3.), and the angle EBD is equal to the angle EGD, for it is subtended by the same arc ED (Prop. XIV. Cor. 2.); therefore the whole angle ABD is equal to the whole angle BGD in the alternate segment.

Again, the angle CBD, together with DBE, make a right angle, also the angle DEB, together with DBE, make a right angle; hence the angle CBD is equal to the angle DEB in the alternate segment.

Cor. 1. *An angle in a segment greater than a semicircle is acute, and an angle in a segment less than a semicircle is obtuse.*

Cor. 2. Therefore *the segment which contains a right angle must be a semicircle, the segment which contains an obtuse angle must be less, and that which contains an acute angle must be greater than a semicircle.*

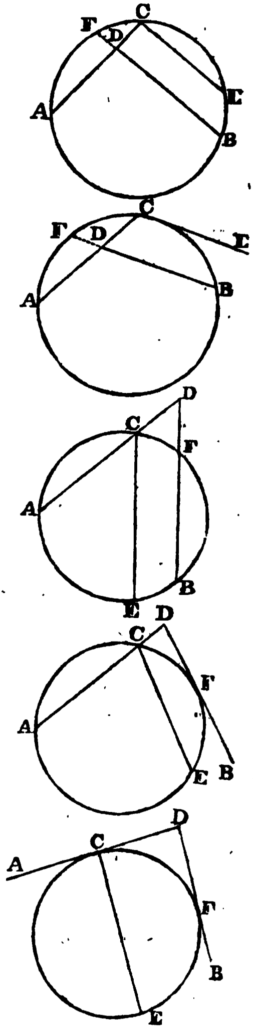
Scholium.

This proposition and the preceding may both evidently be combined in the following general enunciation:—

An inscribed angle, and an angle formed by a tangent and a chord, are each equal to an angle at the centre, subtended by half the arc included by their sides. The following proposition is the converse of this.

PROPOSITION XVI. THEOREM. (*Converse of Props. XIV. and XV.*)

If an angle, whose sides include an arc of a circle, be equal to an angle at the centre whose sides include half that arc, the vertex of the former shall be in the circumference; that is, it shall be either an inscribed angle, or contained by a tangent and a chord.



Let the sides of the angle ADB include the arc AB, and let it be equal to an angle at the centre of the circle, whose sides include half that arc; the point D shall be in the circumference.

For, let D be supposed to lie either within, or without, the circumference, and in the former case let C be the point where the production of one of the sides cuts the circumference, and in the latter case let C be a point where one of the sides meets the circumference, and let CE be drawn parallel to the other side DB.

Then, however the sides of the angle ACE be situated with regard to the circumference; that is to say, whether both are chords, or one a chord and the other a tangent, this angle will in either case be equal to an angle at the centre, subtended by half the arc included by its sides (Prop. XV. Scholium); but, by hypothesis, the angle ADB, which is, by construction, equal to ACE (Prop. XIV. Cor. 1. B. I.), is equal to an angle at the centre, subtended by half the arc included by its sides, but these included arcs are unequal; their halves are, therefore, unequal, and yet they subtend equal angles at the centre, which is impossible (Prop. IV.). The point D, therefore, can lie neither within, nor without, the circumference; it is, therefore, in it.

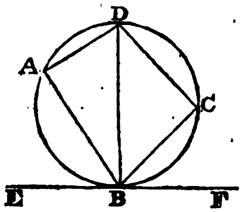
Cor. 1. In the case where D is within the circle, the angle ADB is equal to an inscribed angle C, subtended by an arc, equal to the sum of the arcs AB, CF (Prop. X.), that is, *an angle formed by the intersection of two chords, is equal to an inscribed angle subtended by the* sum *of the opposite intercepted arcs.*

Cor. 2. In the case where D is without the circle, the angle ADB is equal to an inscribed angle, subtended by an arc equal to the difference of the intercepted arcs, that is, *an angle whose vertex is without the circle, and whose sides meet the circumference, is equal to an inscribed angle subtended by the* difference *of the opposite intercepted arcs.*

PROPOSITION XVII. THEOREM.

The opposite angles of a quadrilateral inscribed in a circle, are together equal to two right angles.

The opposite angles A, C, or ABC, ADC, of the inscribed quadrilateral ABCD, are together equal to two right angles.



For let EBF be a tangent to the circle at B, and join BD, then the angle DBF is equal to the angle A, and DBE to the angle C (Prop. XV.); but the angles DBF, DBE are together equal to two right angles: therefore the angles A, C are together equal to two right angles; and since the four angles of the quadrilateral are together equal to four right angles (Prop. XVII. Cor. 1. B. I.), the remaining two must be equal to two right angles.

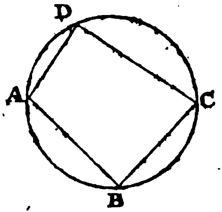
Cor. 1. *If one side of an inscribed quadrilateral be produced, the exterior angle will be equal to the interior opposite one.*

Cor. 2. *A quadrilateral, of which the opposite angles are not equal to two right, cannot be inscribed in a circle.*

PROPOSITION XVIII. THEOREM. (*Converse of Prop. XVII.*)

If the opposite angles of a quadrilateral be together equal to two right angles, a circle may be circumscribed about it.

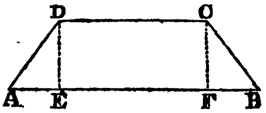
Let ABCD be a quadrilateral, the opposite angles B, D of which are together equal to two right angles, a circle may be circumscribed about it, that is, the circumference which passes through the three points A, B, C, shall also pass through D.



For if D were to lie within the circle, the angle D would be greater than if it were in the circumference (Prop. XVI. Cor. 1.), and consequently the angles B, D would be together greater than two right angles (Prop. XVII.); and if D were to lie without the circle the angle D would be less than if it were in the circumference, and therefore the angles B, D would, in this case, be less than two right angles; D therefore can lie neither within nor without the circle, that is, it is in the circumference.

Cor. 1. *If two opposite angles of a quadrilateral be together equal to the other two opposite angles, a circle may be described about the quadrilateral* (Prop. XVII. Cor. 3. B. I.).

Cor. 2. *A trapezium may be inscribed in a circle, provided the non-parallel sides are equal.*



For if the non-parallel sides AD, BC, of the trapezium ABCD, be equal, and perpendiculars DE, CF, be drawn to AB, the triangles ADE, BCF will be equal, the angle A equal to the angle B, and the angle ADE to the angle BCF, and consequently the angle ADC must be equal to the angle BCD, so that two opposite angles of the trapezium are equal to the other two, therefore a circle may be described about it (Cor. 1.).

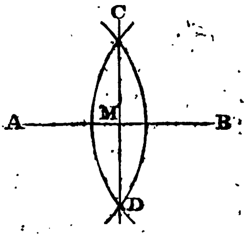
BOOK IV.

POSTULATE.

From any point as a centre with any radius, a circumference may be described.

PROPOSITION I. PROBLEM.

To divide a given straight line AB, into two equal parts.

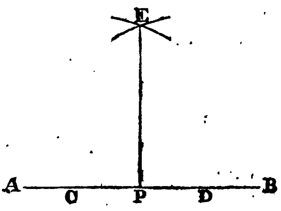


From the points A and B as centres, with any radius greater than half AB, describe two arcs, which must necessarily cut each other (Prop. XIII. Schol. 4. B. III.); draw the straight line CD through the points of intersection, and it will pass through M, the middle of AB; for CD is perpendicular to AB (Prop. XIII. Cor. 3. B. III.), and the points A, B, are equally distant from C, therefore they are equally distant from M (Prop. XXII. Schol. B. I.).

Cor. CD not only divides AB into two equal parts, but it is at the same time perpendicular to AB.

PROPOSITION II. PROBLEM.

From a given point P, in a straight line AB, to draw a perpendicular to that line.

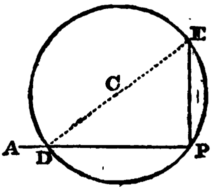


In the given straight line, or in its prolongation, take two points C D, equally distant from P; and from these points as centres, with a radius longer than CP, describe arcs which will intersect in E; draw PE and it will be the perpendicular required; for it is drawn from the middle of the straight line CD to a point equally distant from its extremities (Prop. XXII. Cor. 4. B. I.).

Scholium.

If the point P were the extremity of the line, and if the line could not be produced beyond it, then a different construction must be employed. Thus:

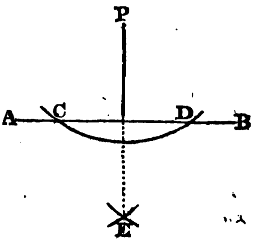
From any point C taken without the line, with a radius equal to the distance CP, describe a circumference, and from D, where it cuts AP, or its prolongation, draw the diameter DE; then EP will be the perpendicular required, as is manifest from Prop. XIV. Cor. 3. B. III.



PROPOSITION III. PROBLEM.

From a given point P, without a straight line AB, to draw a perpendicular to that line.

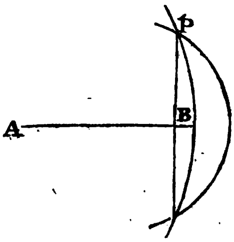
Take any point C in AB, and from P as a centre, with a radius equal to the distance PC, describe an arc CD, and from the points C, D, as centres, with the same, or any other radius, describe two arcs cutting in E, then PE will be the perpendicular required, for it passes through two points P, E, each of which is equally distant from the two extremities of CD (Prop. XXII. Cor. 5. B. I.).



Scholium.

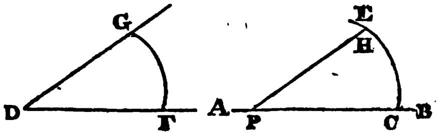
If the point P were opposite the extremity of the line AB, or nearly so, and if AB could not be produced beyond this extremity, the following construction may be employed.

From the other extremity of AB, or from any other point in AB, with a radius equal to its distance from P, describe an arc; then from a second point in AB, with its distance from P as a radius, describe another arc, and through their points of intersection draw a line, which will be the perpendicular required; as is obvious from (Prop. XIII. Cor. 3. B. III.).



PROPOSITION IV. PROBLEM.

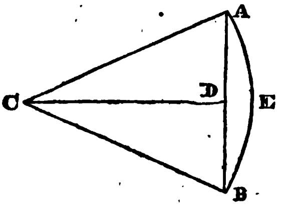
At a point P, in a straight line AB, to make an angle equal to a given angle D.



From P as a centre, with any radius, describe an arc CE, and from D as a centre with the same radius, describe an arc FG, terminating in the sides of the angle D; then, with a radius equal to the chord of this arc, describe, from the point C as a centre, an arc cutting the arc CE in H; draw HP, and HPC will be equal to the angle D; for in equal circles equal arcs subtend equal angles at the centre (Prop. IV. B. III).

PROPOSITION V. PROBLEM.

To divide a given angle, ACB, into two equal parts.



From C as a centre, with any radius, describe an arc AEB, terminating in the sides of the angle, and draw CD perpendicular to its chord; then the angle ACB will be bisected (Prop. V. B. III.).

Scholium.

By repeated bisections an angle may obviously be divided into four, eight, sixteen, &c. equal parts, but the division of an angle into *three* equal parts is a problem that cannot be generally effected by elementary geometry.* This problem is one of very ancient date, and for a long time engaged the attention of some of the greatest geometers of Greece, who were at length compelled to relinquish the hope of performing this operation by a purely geometrical method, that is to say, by employing no other lines in the construction of the problem than the straight line and the circumference of a circle. By the introduction of other curves, the trisection has been effected in various ways.

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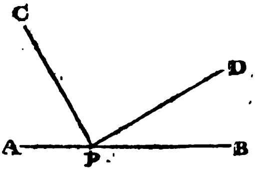
* The trisection of a *right angle*, and hence of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. thereof, is a very easy problem, the construction of which may be left for the student to perform. It may not be improper to remark here, that an ingenious instrument, for the mechanical trisection of angles, has recently been devised by Mr. R. Christie, for a description of which see Mechanic's Magazine, vol. I.

PROPOSITION VI. PROBLEM.

Two angles of a triangle being given, to find the third angle.

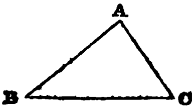


Draw any straight line AB, and take therein a point P, at which make an angle APC equal to one of the given angles, and then another CPD, equal to the other given angle: the third angle DPB will be equal to the third angle of the triangle, for the three angles of the triangle are together equal to the three angles at the point P, each amounting to two right angles, and two of the angles at P have been made equal to two angles of the triangle, therefore the third angle must be equal to the remaining angle of the triangle.

PROPOSITION VII. PROBLEM.

Two angles and a side of a triangle being given, to construct the triangle.

First, let the angles be adjacent to the given side.



Draw the straight line BC, equal to the given side, and at the extremities make two angles, BCA, CBA, equal to those given, then the sides BA, CA, must meet and form with BC the triangle required; for if they were parallel, the angles B, C, would be together equal to two right angles (Prop. XIV. Cor. 4. B. I.), and therefore could not belong to a triangle.

But if one of the given angles be opposite to the given side, then find the other angle by last proposition, and proceed as above.

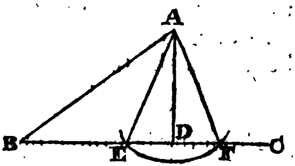
PROPOSITION VIII. PROBLEM.

Two sides of a triangle, and the angle which they include being given, to construct the triangle.

Draw BC (preceding diagram) equal to the given side, make an angle CBA equal to the given angle, and take BA equal to the other given side: join AC, and the triangle will evidently be constructed.

PROPOSITION IX. PROBLEM.

Two sides of a triangle, and an angle opposite to one of them being given, to construct the triangle.



At the extremity B, of any straight line BC, make an angle CBA equal to the given angle, and make BA equal to that side which is adjacent to the given angle; and from A, as a centre, with a radius equal to the other side, describe an arc, which must either touch, or cut, the line BC, otherwise a triangle could not be formed. If it touch BC, a line from A to the point of contact D, will be perpendicular to BC (Prop. IX. B. III.), and the right angled triangle ABD will be the triangle required.

The given angle in this case must, therefore, be *acute*.

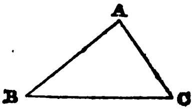
But, if instead of touching, the arc cuts BC in two points E, F; then, supposing still that the given angle is acute, if lines be drawn from these points to A, it is obvious that two triangles ABE, ABF, will be formed, each of which will contain the proposed given parts; but will differ in other respects, the angle opposite AB being obtuse in the one, and acute in the other. In order, therefore, to avoid this ambiguity, it is requisite previously to know whether the angle opposite the other given side be acute or obtuse.

If, however, the given angle be *obtuse*, no ambiguity can arise; for having formed the obtuse angle as BEA, and having made EA equal to the adjacent side, the arc described from A as a centre, with a radius equal to AB, the other side, would cut BC on opposite sides of E (Prop. XXII. Schol. B. I.); so that only one obtuse angled triangle could be formed.

And if the given angle were right, although two triangles would be formed, yet, as the hypothenuses would meet BC at equal distances from the common perpendicular, these triangles would be equal.

PROPOSITION X. PROBLEM.

The three sides of a triangle being given, to construct it.



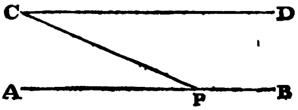
Make BC equal to one of the sides, and from B, as a centre, with one of the other sides as a radius, describe an arc; and from C, as a centre, with a radius equal to the third side, describe another arc, cutting the former in a point A (Prop. XXI. B. I., and Prop. XIII. Schol. 4. B. III.); then, if AB, AC, be drawn, the triangle will be constructed.

Scholium.

From the last three problems it appears that, of the six parts composing a triangle, viz. the sides and the angles, it is necessary to know but three, and their relative positions, in order to determine the triangle. It is, however, requisite that at least one of the given parts be a side, and, moreover, in the case where two sides and an opposite acute angle are the given parts, it is indispensable to know, in order to avoid ambiguity, whether the angle opposite the other given side be acute or obtuse.

PROPOSITION XI. PROBLEM.

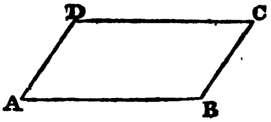
Through a given point C, to draw a straight line parallel to a given straight line AB.



To any point P in AB draw a straight line from C, then make the angle PCD equal to the angle APC, and CD will be parallel to AB (Prop. XII. B. I.).

PROPOSITION XII. PROBLEM.

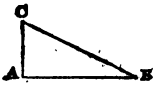
Two adjacent sides of a rhomboid, with the angle which they include being given, to construct the rhomboid.



Make AB equal to one of the given sides, and the angle A equal to the given angle; then take AD equal to the other given side, and from D, as a centre, with a radius equal to AB, describe an arc; and from B, as a centre, with a radius equal to AD, describe another; and from C, the point where they intersect, draw CB, CD, and the rhomboid will be completed; for the opposite sides are, by construction, equal (Prop. XXVIII. B. I.).

PROPOSITION XIII. PROBLEM.

To make a square equivalent to two given squares.



Draw two indefinite lines AB, AC, perpendicular to each other. Take AB equal to the side of one of the given squares, and AC equal to the side of the other: join BC, which will be the side of the proposed square, as is evident from Prop. X. B. II.

Scholium.

Any number of squares may be reduced to a single one, by reducing two into one, this and a third into another; then, again, this last and a fourth into another, and so on.

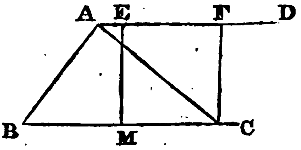
PROPOSITION XIV. PROBLEM.

To make a square equivalent to the difference of two given squares.

Draw, as in last problem, the lines AB, AC (see the diagram) perpendicular to each other, making AB equal to the side of the less square; then, from B as a centre, with a radius equal to the side of the other square, describe an arc intersecting AC in C, and AC will be the side of the required square (Prop. X. Cor. 1. B. II.).

PROPOSITION XV. PROBLEM.

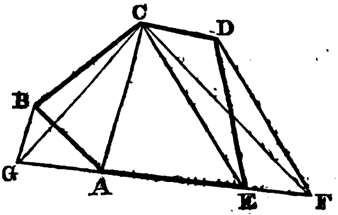
To make a rectangle equivalent to a given triangle, ABC.



Draw AD parallel to the base BC, which bisect by the perpendicular ME, make EF equal to MC; then, by drawing FC, the rectangle EC equal to the triangle ABC will be formed; for it has the same altitude ME as the triangle, and half its base (Prop. III. Cor. 5. B. II.).

PROPOSITION XVI. PROBLEM.

To make a triangle equivalent to any given polygon, ABCDE.

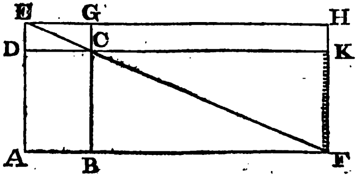


Draw the diagonal CE, cutting off the triangle CDE; draw DF parallel to CE, meeting AE produced, and join CF: the polygon ABCDE will thus be reduced to the polygon ABCF, having fewer sides by one; for the triangle CDE cut off, is equivalent to the triangle CFE added (Prop. III. Cor. 2. B. II.). Draw, now, the diagonal CA and BG parallel to it, meeting the production of EA: join CG, and the polygon ABCF will be reduced to an equivalent polygon, with fewer sides by one; for the triangle, cut off by CA, has been supplied by the equivalent triangle CGA: and thus, by continually diminishing the sides, the polygon is at length reduced to an equivalent triangle.

Cor. Since a triangle may be converted into an equivalent rectangle, it follows that *any polygon may be reduced to an equivalent rectangle.*

PROPOSITION XVII. PROBLEM.

A rectangle being given, to construct an equivalent one having a side of a given length.



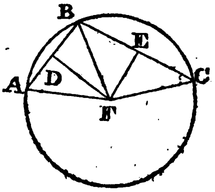
Let ABCD be the given rectangle, and produce one of its sides, as AD, till DE be the given length, and draw ECF meeting the prolongation of AB in F; then produce BC till CG is equal to DE; draw EGH, DCK, making GH, CK, each equal to BF; then join HK, and the rectangle GK will be equivalent to the rectangle AC, as appears from Proposition I. Book II.

Cor. Hence *a polygon may be converted into an equivalent rectangle of a given base, or of a given altitude* (Prop. XVI. Cor.).

PROPOSITION XVIII. PROBLEM.

Having given a circumference, or an arc, to find the centre of the circle.

Take any three points A, B, C, in the arc, bisect the distances AB, BC, by the perpendiculars DF, EF (Prop. I.), these perpendiculars will meet in a point F equally distant from the points A, B, C (Prop. VIII. B. III.); F, therefore, is the centre of the circle.



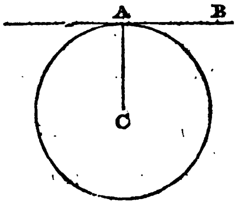
Scholium.

By a similar construction a circumference may be circumscribed about a given triangle. If the triangle be right angled, the middle of the hypothenuse will be the centre of the circumscribed circle.

PROPOSITION XIX. PROBLEM.

To draw a tangent to a circle from a given point A, in the circumference.

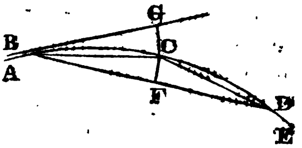
Draw the radius CA, and make AB perpendicular to it; AB will be the tangent required (Prop. IX. B. III.).



PROPOSITION XX. PROBLEM.

From a given point B, in the arc ABE, of a circle, to draw a tangent thereto, without making use of the centre.

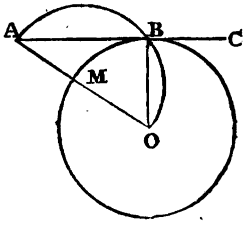
Take two equal distances BC, CD; on the arc join BD, and from B as a centre, with a radius equal to the distance BC, describe an arc FG; make the distance CG equal to the distance CF, and through G draw the straight line BG, which will be the tangent required; for if the



chords BC, CD be drawn, the angle CBD will be equal to the angle CDB (Prop. XIV. Cor. 2. B. III.); and therefore the angle GBC will be equal to the angle CBF, that is, to CDB, an angle in the alternate segment; hence BG is a tangent at B.

PROPOSITION XXI. PROBLEM.

To draw a tangent to a circle from a given point A, without the circumference.



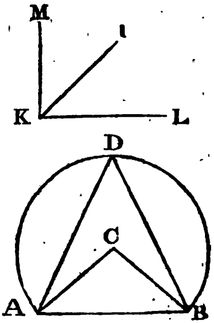
Bisect the distance AO between the given point and the centre, in the point M, and from this point as a centre, with a radius equal to MA, describe an arc; and through B, where it intersects the circle, draw ABC, which will be the tangent required for completing the semicircle ABO, and drawing BO, the angle ABO will be right (Prop. XIV. Cor. 3. B. III.), that is, AB is perpendicular to BO, and is therefore a tangent to the circle at B (Prop. IX. B. III.).

Scholium.

Since the semicircle described on the other side of AO intersects the given circumference in another point, it follows that two tangents may be drawn from the same point.

PROPOSITION XXII. PROBLEM.

On a given straight line AB, to describe a segment capable of containing a given angle, IKL.



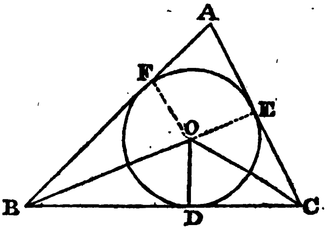
Draw KM perpendicular to KL, one of the sides of the given angle; then, at each extremity of AB, make an angle equal to IKM, the sides AC, BC, of which will meet in a point C, from which, as a centre, with a radius equal to CA, or CB, describe the arc ADB, and the required segment will be completed; for the three angles of the triangle ABC are together equal to twice the angle MKL, and, by construction, the two angles CAB, CBA, are together equal to twice IKM; therefore the angle C is equal to twice IKL, but it is also equal to twice the inscribed angle D: consequently the segment ADB contains an angle equal to the given angle IKL.

Scholium.

If the given angle were obtuse, the centre must lie without the segment (Prop. XV. Cor. 2. B. III.); if it were right, the segment would be a semicircle; and, consequently, the centre would be the middle of AB.

PROPOSITION XXIII. PROBLEM.

To inscribe a circle in a given triangle, ABC.



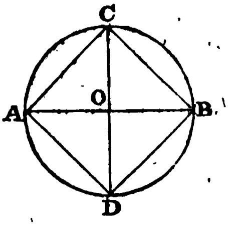
Bisect two of the angles, as B, and C, by the straight lines BO, CO; from the point of meeting, draw OD perpendicular to BC, and from O as a centre, with a radius equal to OD, describe the circle DEF, which will touch each side of the triangle; for, let OE, OF, perpendiculars to the other two sides, be drawn, then the right angled triangles ODC, OEC, have the hypothenuses equal, as also the angles at C; therefore OD is equal to OE (Prop. XI. B. I.); for similar reasons OD is equal to OF: hence the circumference described from the centre O, with the radius OD, touches the sides of the triangle in D, E, and F.

Scholium.

Lines bisecting the three angles of a triangle, all meet in the same point, viz. the centre of the inscribed circle; and it has been shown, Prop. XVIII. that the lines bisecting the three sides also meet in one point; consequently in the equilateral triangle, since the lines bisecting the angles also bisect the sides (Prop. IX. Cor. 1. B. I.), it follows that the centres of the inscribed and circumscribed circles coincide.

PROPOSITION XXIV. PROBLEM.

To inscribe a square in a given circle.



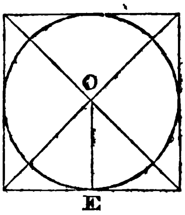
Draw two diameters AB, CD, at right angles to each other, then join their extremities, and the inscribed square will be formed; for the angle ACB, being in a semicircle, is right, and the angles about O being equal, the chords which subtend them are equal (Prop. III. Cor. 1. B. III.).

Scholium.

By a similar construction it is plain that a circle may be circumscribed about a given square.

PROPOSITION XXV. PROBLEM.

To inscribe a circle in a given square.



From the point O, where the diagonals intersect, draw OE perpendicular to a side of the square, and then, from O as a centre, with a radius equal to OE, describe a circle which will touch each side of the square; for the square is divided by the diagonals into four equal isosceles triangles: hence the perpendicular, from the vertex O to the base, is the same in each triangle (Prop. III. Cor. 6. B. II.); therefore the circumference described from the centre O with the radius OE, passes through the extremities of each perpendicular; so that the sides of the square are tangents to the circle (Prop. IX. B. III.).

Cor. Hence, *from a common centre a circle may be inscribed in, and circumscribed about, a given square* (Prop. XXIV. Schol.).

BOOK V.

DEFINITIONS.

1. Of two unequal magnitudes, the greater is said to *contain* the less, as often as there are parts in the greater equal to the less.

2. If the greater of two magnitudes contain the less a certain number of times without leaving a remainder, it is called a *multiple* of the less; and the less is, in this case, called a *submultiple*, or a *measure* of the greater.

3. Magnitudes, which have a *common measure*, are said to be *commensurable;*—they are *incommensurable* when such common measure does not exist.

4. *Equimultiples*, or *like multiples*, are those which contain their respective submultiples the same number of times.

5. And *like submultiples* are those which are contained in their respective multiples the same number of times.

6. When two magnitudes are compared, by examining how often the first is contained in a multiple of the second, the former is called an *antecedent*, and the latter its *consequent*. Such comparison, it is obvious, can be made only between *homogeneous* magnitudes, or those of the same kind.

7. Magnitudes are *proportional* when an antecedent cannot be contained in any multiple of its consequent oftener than either of the other antecedents can be contained in a like multiple of its consequent.

8. If the number of magnitudes so related be but four, they are denominated simply a *proportion;* the first and last terms are called the *extremes*, and the intermediate terms the *means*.

9. The antecedents are called *homologous*, or *like* terms, and the same of the consequents.

10. Magnitudes are in *continued proportion* when every consequent is considered as the antecedent of the succeeding term.

11. If the continued proportion consist of but three terms, the middle term is called the *mean*, and the others are called the *extremes*.

Explanation of the Signs employed in the succeeding Demonstrations.

1. To denote that four magnitudes A, B, C, D, are proportional, they are thus arranged, $A : B :: C : D$, which is read, *as A is to B, so is C to D*, understanding, of course, that relation among them which constitutes their proportionality (Def. 7.).

2. In like manner, to denote a continued proportion, the terms are thus expressed, $A : B :: B : C :: C : D$, &c., which is read, *as A is to B, so is B to C, and C to D, &c.*

3. The symbol $+$ denotes the *addition* of the magnitudes between which it is placed. Thus $A+B$ signifies that B *is to be added to* A. To denote *subtraction*, the sign $-$ is employed; so that $A-B$ means B *taken from* A. The double sign $\pm$ is employed to express indifferently either the *sum or difference* of the magnitudes between which it is placed.

4. The symbol $>$ placed between two magnitudes denotes that the former is *greater* than the latter; the symbol $<$ is used to denote that the former is *less* than the latter, and $=$ implies their *equality*. Thus by $A > B$ is to be understood that A *is greater than* B; by $A < B$ is implied that A *is less than* B, and by $A = B$ is meant that A *is equal to B.*

Note.—It is to be observed that, in speaking of the magnitudes A, B, C, &c., we mean, in reality, those which these letters are employed to represent; they may be either lines, surfaces, or solids.

AXIOMS.

1. Equimultiples of the same, or of equal magnitudes, are equal, so also are the like submultiples.

2. A multiple of a greater magnitude exceeds a like multiple of a less; and a submultiple of a greater exceeds a like submultiple of the less.

3. Of two unequal magnitudes, a multiple of the less may be taken so great as to exceed the greater, and a submultiple of the greater may be taken so small as to be smaller than the less.

PROPOSITION I. THEOREM.

If two magnitudes be like multiples of two others, the sum of the former will be the same multiple of the sum of the latter.

Let P, Q be equimultiples of A, B, then will $P+Q$ be a like multiple of $A+B$.

A B
P Q.

For, suppose P to be divided into parts, each equal to A, and Q to be divided into parts, each equal to B; then, by hypothesis, the number of the parts of P is the same as the number of the parts of Q. Let one of the parts of P be added to one of the parts of Q, the sum will be equal to $A+B$, and if any other part of P be added to any other of the parts of Q, the sum will, in like manner, be equal to $A+B$; therefore, as many magnitudes as there are in P equal to A, or as there are in Q equal to B, so many are there in $P+Q$ equal to $A+B$.

Scholium.

Having proved it of two magnitudes, it may be proved of three, and so on of any number.

PROPOSITION II. THEOREM.

If, in a proportion, an antecedent and its consequent be respectively the same as an antecedent and its consequent in another proportion, the remaining antecedent and consequent in each will form a proportion.

Let there be the two proportions,

$$A : B :: C : D,$$
$$A : B :: E : F.$$

Then C is contained in any multiple of D, as often as A is contained in a like multiple of B, but not oftener (Def. 7.). In like manner E is contained in any multiple of F, as often as A is contained in the same multiple of B, but not oftener: hence, C cannot be contained oftener in any multiple of D than E is contained in a like multiple of F, nor can E be contained oftener in any multiple of F than C is contained in a like multiple of D; consequently, (Def. 7.)

$$C : D :: E : F.$$

Scholium.

Two or more proportions, having an antecedent and its consequent the same in each, may therefore be combined in a single series of proportionals: thus the two proportions above given form the following series,

A : B :: C : D :: E : F.

This method of combining the terms of such proportions in a single series will often be employed hereafter, as it will be unnecessary to repeat the identical terms.

PROPOSITION III. THEOREM.

In a proportion, it will be impossible to find equimultiples of the antecedents and equimultiples of the consequents, such that the multiple of one antecedent may be greater than that of its consequent, while the multiple of the other antecedent is not greater than that of its consequent.

For, let there be the proportion A : B :: C : D, and, if the truth of the theorem be denied, suppose P, R to be equimultiples of A, C, and Q, S equimultiples of B, D; such that P is greater than Q, and R not greater than S.

A	B	C	D
P	Q	R	S

Then since, by hypothesis, C is the same submultiple of R that A is of P, and since S is not less than R, C is necessarily contained in S as often, at least, as A is contained in P, and, therefore, oftener than A is contained in Q, for Q is less than P; that is to say, the antecedent C is contained oftener in S, a multiple of its consequent, than the other antecedent A is contained in Q, a like multiple of its consequent, which is impossible (Def. 7.); therefore if P be greater than Q, R must be greater than S.

PROPOSITION IV. THEOREM.

If there be four magnitudes, such that it is impossible to find equimultiples of the antecedents and equimultiples of the consequents, such that the multiple of one antecedent may be greater than that of its consequent, while the multiple of the other antecedent is not greater than that of its consequent, the four magnitudes are proportional.

Let A, B, C, D, be four such magnitudes, then, if they are not proportional, one of the antecedents, as A, must be contained in some multiple Q of its consequent B, oftener than C is contained in S, a like multiple of D (Def. 7.).

A B C D
P Q R S

Let P be the greatest multiple of A which does not exceed Q, and let R be a like multiple of C.

Then R must be greater than S, for R contains C as often as Q contains A, which, by hypothesis, is oftener than S contains C; so that equimultiples P, R of the antecedents, and equimultiples Q, S of the consequents may be found, such that R shall be greater than S, while P is not greater than Q, which contradicts the hypothesis.

PROPOSITION V. THEOREM.

If any number of homogeneous magnitudes be proportional, as one antecedent is to its consequent, so is the sum of the antecedents to the sum of the consequents.

First, let there be four magnitudes, or the proportion A : B : : C : D, then also A : B : : A + C : B + D.

For let P, R be equimultiples of A, C, and Q, S equimultiples of B, D.

A B C D
P Q R S

Then (Prop. III.), if $P > Q$, $R > S$, or if $R > S$, $P > Q$; therefore if $P > Q$, $(P + R) > (Q + S)$, and if $(P + R) > (Q + S)$, $P > Q$; for if, in this last case, P were not greater than Q, R could not be greater than S; and, therefore, $P + R$ could not be greater than $Q + S$. Now, P and $(P + R)$ are any equimultiples of A and $(A + C)$ (Prop. I.), in like manner Q and $(Q + S)$ are any equimultiples of B and $(B + D)$; therefore (Prop. IV.)

$$A : B :: A + C : B + D.$$

Let there be six magnitudes, A : B : : C : D : : E : F; then, with respect to the first four, there will be the proportion A : B : : A + C : B + D, while the last four furnish the proportion C : D : : C + E : D + F, but A : B : : C : D; therefore (Prop. II.) A : B : : C + E : D + F; hence, from what has been already demonstrated,

$$A : B :: A + C + E : B + D + F,$$

and so on for any number of proportionals.

Cor. 1. Since $A : B :: A : B :: A : B$, &c., it follows that $A : B :: A + A + A +$ &c. $: B + B + B +$ &c., that is, *two magnitudes and their like multiples are proportional.*

Cor. 2. Hence also *two magnitudes and their like submultiples are proportional.*

Cor. 3. Wherefore, *in any proportion, one antecedent is to its consequent as any multiple or submultiple of the other antecedent is to a like multiple or submultiple of its consequent* (Prop. II.).

Cor. 4. And moreover, *if in any proportion like multiples or like submultiples of either the two first, or the two last terms be taken, and like multiples or submultiples of the others, the results will be proportional.*

PROPOSITION VI. THEOREM.

If in any proportion like multiples of the antecedents and like multiples of the consequents be taken, the results will be proportional.

In the proportion $A : B :: C : D$, let P, R be equimultiples of A, C, and Q, S equimultiples of B, D; then $P : Q :: R : S$.

For let P', R' be any equimultiples of P, R; and Q', S' any equimultiples of Q, S;

$$\begin{matrix} P & Q & R & S \\ P' & Q' & R' & S' \end{matrix}$$

Then it is obvious that P', R' must be equimultiples of A, C, and Q' S' equimultiples of B, D; therefore (Prop. III.) if $P' > Q'$, then $R' > S'$, and if $R' > S'$, then $P' > Q'$, and P', R' are any equimultiples of P, R, while Q', S are any equimultiples of Q, S; consequently (Prop. IV.),

$$P : Q :: R : S.$$

Cor. 1. *In any proportion the first term is to any multiple of the second as the third is to a like multiple of the fourth.*

For, as above, let P, R be any equimultiples of A, C; and Q, S any equimultiples of B, D; while Q', S are any equimultiples of Q, S; these last will obviously be equimultiples of B, D, and consequently (Prop. III.) if $P > Q'$ then $R > S$, and if $R > S$ then $P > Q$, and P, R are any equimultiples of A, C, while Q', S' are any equimultiples of Q, S; therefore (Prop. IV.)

$$A : Q :: C : S.$$

Cor. 2. It follows moreover that *any submultiple of the first term is to the second as a like submultiple of the third is to the* *fourth*, for, in the last proportion, Q, S are any equimultiples of B, D; and if *P*, *R* be the same submultiples of A, C, we have, by Cor. 4. Prop. V.

$$P : B :: R : D.$$

PROPOSITION VII. THEOREM.

If in a proportion, an antecedent be either greater or less than its consequent, the other antecedent will, in like manner, be either greater or less than its consequent.

Let the proportion be A : B :: C : D; and suppose first that $A > B$; then also $C > D$.

For let Q, S be any equimultiples of B, D.

$$\begin{array}{cccc} A & B & C & D \\ & Q & & S \end{array}$$

Then because $A > B$, Q contains B oftener than it contains A; and, since S contains D as often as Q contains B, it follows that S contains D oftener than Q contains A; but (Def. 7.) Q contains A as often as S contains C, therefore S contains D oftener than it contains C, and consequently $C > D$.

Next let $A < B$, then also $C < D$.

For, whatever be the difference between A and B, a multiple thereof may be taken, so great as to exceed A; and if the same multiple of B be taken, it must evidently contain A oftener than it contains B. Let Q be this multiple of B, and let S be an equimultiple of D. Then since S contains C as often as Q contains A, S must contain C oftener than Q contains B; but Q contains B as often as S contains D, consequently S contains C oftener than it contains D; hence $C < D$.

Cor. Therefore *if one antecedent be equal to its consequent, the other antecedent will be equal to its consequent.*

PROPOSITION VIII. THEOREM.

The terms of a proportion are proportional when taken inversely, that is, as the second is to the first, so is the fourth to the third.

Let the proportion be A : B :: C : D, then also B : A :: D : C.

For let P, R be any equimultiples of A, C, and Q, S any equimultiples of B, D.

B A D C
Q P S R

Then (Prop. VI.) P : Q :: R : S; therefore (Prop. VII.), if Q > P, then S > R, and if S > R, then Q > P, consequently, (Prop. IV.)

B : A :: D : C.

Cor. 1. *In any proportion, a multiple of the first term is to the second, as a like multiple of the third term is to the fourth* (Cor. 1. Prop. VI.); *also the first term is to a submultiple of the second as the third is to a like submultiple of the fourth.*

Cor. 2. It follows from this and Cor. 2. Prop. VI. that *if in a proportion like submultiples of the antecedents and like submultiples of the consequents be taken, the results will be proportional.*

PROPOSITION IX. THEOREM.

In a proportion consisting of homogeneous magnitudes, if one antecedent be greater than the other, the consequent of the former will be greater than the consequent of the latter.

In the proportion A : B :: C : D, let A > C, then also B > D.

By inversion (Prop. VIII.) B : A :: D : C, and whatever be the difference between A and C, it is possible for a multiple thereof to exceed D, and consequently such a multiple of A must contain D oftener than an equimultiple of C. Let P, R be these equimultiples of A, C.

B A D C
P R.

Then (Def. 7.) P does not contain B oftener than R contains D, but P does contain D oftener than R contains it; consequently P contains D oftener than it contains B; therefore B > D.

Cor. 1. It follows by inversion, that *if one consequent be greater than the other, the antecedent of the former will be greater than that of the latter.*

Cor. 2. Consequently *if the antecedents be equal, the consequents will be equal, and if the consequents be equal the antecedents will be equal.*

Cor. 3. Hence, and from Prop. II. *if in two proportions there be three corresponding terms in each respectively equal, the fourth terms will be equal.*

PROPOSITION X. THEOREM.

If the terms of a proportion are all of the same kind, they are proportional when taken alternately, that is, as the first is to the third, so is the second to the fourth.

Let the proportion be A : B :: C : D, then also A : C :: B : D.

For let P, Q be any equimultiples of A, B, and R, S any equimultiples of C, D.

A C B D
P R Q S

Then (Prop. V. Cor. 4.) P : Q :: R : S; therefore, (Prop. IX.) if $P > R$, then $Q > S$, and if $Q > S$, then $P > R$; consequently (Prop. IV.)

A : C :: B : D.

Cor. 1. Hence, and from Cor. 3. to Proposition V. it follows that, *in such a proportion, the first term is to the third as any multiple or submultiple of the second to a like multiple or submultiple of the fourth.*

Cor. 2. Likewise (Cor. 4. Prop. V.) *like multiples or like submultiples of the first and third terms are to each other as like multiples or submultiples of the second and fourth terms.*

PROPOSITION XI. THEOREM.

If, in a proportion, an antecedent be a multiple or submultiple of its consequent, the other antecedent will be a like multiple or submultiple of its consequent.

In the proportion A : B :: C : D, let A be a multiple of B, then C will be the same multiple of D.

Take Q equal to A, and let S be the same multiple of D that Q or A is of B.

A B C D
Q S

Then (Prop. VI. Cor.) A : Q :: C : S, but A is equal to Q; therefore (Prop. VII. Cor.) C is equal to S, but S is the same multiple of D that A is of B; therefore C is the same multiple of D that A is of B.

Again, let A be a submultiple of B, then will C be a like submultiple of D; for, by inversion (Prop. VIII.), B : A :: D : C; therefore, as just shown, D is the same multiple of C that B is of A: in other words, C is the same submultiple of D that A is of B.

Cor. Therefore, when the terms are homogeneous, *if one antecedent be a multiple or submultiple of the other, the consequent of the former will be a like multiple or submultiple of the consequent of the latter* (Prop. X.).

PROPOSITION XII. THEOREM.

In a proportion, the sum of an antecedent and its consequent is to either term, as the sum of the other antecedent and consequent to the like term.

Let the proportion be A : B :: C : D.

Then A cannot be contained oftener in any multiple of B, than C is contained in a like multiple of D, (Def. 7.); therefore A cannot be contained oftener in any multiple of A+B, than C is contained in a like multiple of C+D; hence A : A+B :: C : C+D. Again, by Prop. VIII., B : A :: D : C; therefore, as just proved, B : B+A :: D : D+C.

Cor. 1. Hence (Prop. VIII.), A+B : A :: C+D : C, and A+B : B :: C+D : D.

Cor. 2. And (Prop. X.) when the terms are homogeneous A+B : C+D :: A : C, or A+B : C+D :: B : D.

PROPOSITION XIII. THEOREM.

In a proportion, the difference between an antecedent and its consequent is, to either term, as the difference between the other antecedent and consequent to the like term.

In the proportion A : B :: C : D, let B be greater than A, and D greater than C (Prop. V.), and let $B-A=B'$, and $D-C=D'$.

Then A cannot be contained oftener in any multiple of B′ than C is contained in a like multiple of D′: for if it could, A would obviously be contained oftener in the same multiple of B, than C is contained in a like multiple of D, which is impossible (Def. 7.); in like manner C cannot be contained oftener in a multiple of D′ than A is contained in a like multiple of B′; therefore (Prop. IV.), A : B′ :: C : D′, that is,

$$A : B-A :: C : D-C$$

and (Prop. XII.) B : B−A :: D : D−C.

Again, let A be greater than B, then (Prop. VIII.) B : A :: D : C; therefore, as has just been proved, B : A−B :: D : C−D; and A : A−B :: C : C−D.

PROPOSITION XIV. THEOREM.

In a proportion, the sum of the greatest and least terms exceeds the sum of the other two.

Let the proportion be A : B :: C : D, and let A be the greatest term, then D will be the least (Props. VII. and IX.) and $(A+D)>(B+C.)$

For, by the preceding proposition, $A : A-B :: C : C-D$, and alternately, $A : C :: A-B : C-D$, and A being greater than C, $A-B$ is greater than $C-D$ (Prop. V.); therefore if $B+D$ be added to each, there will result $(A+D)>(B+C)$.

If B be the greatest term, then, by inversion, B : A :: D : C, so that C is the least term, and, by reasoning as above, there will result $(B+C)>(A+D)$.

Cor. In the proportion A : B :: B : C, $(A+C)>2B$, so that *the mean term of three proportionals is less than half the sum of the extremes.*

PROPOSITION XV. THEOREM.

If there be two series of magnitudes, such that the first term is to the second in the first series as the first term is to the second in the other series, and the second term to the third in the former as the second to the third in the latter, and so on; then as the first term is to the last in the one series, so is the first to the last in the other.

First, let there be three magnitudes in each series, viz. A, B, C in the one, and D, E, F in the other, furnishing the two proportions A : B :: D : E, and B : C :: E : F; then also

A : C :: D : F.

For take P, R any equimultiples of A, D; and Q, S any equimultiples of C, F.

A C D F
P Q R S.

Then it is to be proved that if $P>Q$, we must have $R>S$; and if $R>S$, then also $P>Q$ (Prop. IV.). Let us suppose $P>Q$.

Since P, R are equimultiples of A, D, and Q, S equimultiples of C, F, it follows (Prop. VIII. Cor. 1.) that

P : B :: R : E, and Q : B :: S : E.

Let M be a submultiple of B less than $P-Q$, and take N an equisubmultiple of E, then M must be contained oftener in P than in Q. Now (Prop. VIII. Cor. 1, and by inversion),

$$M : P :: N : R, \text{ and } M : Q :: N : S;$$

therefore (Def. 7.) N is contained as often in R as M is contained in P, and consequently N is contained in R oftener than M is contained in Q; but (Def. 7.) M is contained in Q as often as N is contained in S, therefore N is contained in R oftener than N is contained in S; consequently $R > S$. In a similar manner, if we had supposed $R > S$, it would have resulted that $P > Q$; consequently (Prop. 4.)

$$A : C :: D : F.$$

Next let there be three magnitudes in each series, viz.: A, B, C, D, and E, F, G, H, furnishing the additional proportion $C : D :: G : H$.

Then, as shown above, $A : C :: E : G$, and since $C : D :: G : H$, it follows by the preceding case, that $A : D :: E : H$, and so on for any number of magnitudes.

Cor. 1. *If the consequents in one proportion be the antecedents in another, a third proportion may be formed with the same antecedents as the first proportion and the same consequents as the second.*

Cor. 2. Also *if the antecedents in two proportions be the same, the consequents of the one are as the consequents of the other, each to each; or if the consequents be the same, then the antecedents of the one are as those of the other, each to each.* This immediately follows from last corollary, by inverting the terms of the proportions.

Cor. 3. From Cor. 1. Prop. XII. it appears, that if $A : B :: C : D$, then $A+B : A :: C+D : C$; and from Prop. XIII. $A : A-B :: C : C-D$; hence

$$A+B : A-B :: C+D : C-D, \text{ or, when } B > A,$$
$$A+B : B-A :: C+D : D-C.$$

PROPOSITION XVI. THEOREM.

If the antecedents in one proportion be the same as those in another, then the first antecedent is to the sum or difference of the first consequents, as the second antecedent is to the sum or difference of the second consequents.

Let the proportions be $A : B :: C : D$ and $A : E :: C : F$, then, also, $A : B \pm E :: C : D \pm F$.

For (Cor. 2. last Prop.) B : E :: D : F, therefore (Props. XII. and XIII.) B : B±E :: D : D±F; but from the first of the proposed proportions B : A :: D : C; hence (Prop. XV. Cor. 2.) A : B±E :: C : D±F.

Cor. 1. It is likewise obvious that this result, combined with the first of the proposed proportions, gives (Cor. 2. last Prop.)

$$B : B \pm E :: D : D \pm F.$$

Cor. 2. If the terms of both proportions are all homogeneous, then, by alternation (Prop. X.)

$$A : C :: B \pm E : D \pm F, \text{ and } B : D :: B \pm E : D \pm F.$$

Scholium.

It is obvious that the reasoning might be extended to three proportions, then to four, and so on to any number.

PROPOSITION XVII. THEOREM.

If a magnitude measure each of two others, it will also measure their sum and difference.

Let C measure A, or be contained in it a certain number of times exactly; 5 times for instance: let C be also contained in B, suppose 9 times. Then A is equal to 5 times C, and B is equal to 9 times C: consequently A and B together must be equal to 14 times C, so that C measures the sum of A and B; likewise, since the difference of A and B is equal to 4 times C, C also measures this difference, and had any other numbers been chosen, it is plain that the results would have been similar.

Cor. If C measure B, and also A−B, or A+B, it must measure A, for the sum of B and A−B is A, and the difference of B and A+B is also A.

PROPOSITION XVIII. PROBLEM.

Two magnitudes of the same kind being given, to find their greatest common measure.

Let A and B be the proposed magnitudes, it is required to find the greatest magnitude that can measure them both.

Suppose A to be the greater of the two magnitudes, and let B be taken as often as possible from A, leaving a remainder C less than B; let C be in like manner taken as often as possible from B, leaving a remainder D less than C; now let D be in a similar way taken as often as possible from C, leaving a remainder E less than D, and so on till no remainder be left, which will be the case when the last measures the preceding one. The last remainder will be the greatest common measure sought.

For the magnitude sought, as it measures B, will measure any multiple thereof, and consequently, since it also measures A, it must measure the difference between any multiple of B and A (Prop. XVII.); but C is the difference between a multiple of B and A, therefore it measures C. Again, since it must measure a multiple of C, it must measure likewise the difference between a multiple of C and B (Prop. XVII.); it must therefore measure D; in a similar manner, by continuing this reasoning, is it to be shown that the required measure must also measure E, and so on as long as there are any remainders.

Now let us suppose that E is the last remainder; then E measures D, and therefore any multiple of it; and since the difference between C and a multiple of D is equal to E, E must measure C (Prop. XVII. Cor.), it must therefore measure a multiple of C, and since the difference between B and a multiple of C is equal to D, which E has already been shown to measure, E must measure B; it must therefore measure a multiple of B; and since the difference between A and a multiple of B is C, which E has been shown to measure, E must measure A; E therefore measures both A and B. Now it has been shown above that every common measure of A and B measures the last remainder, and we have just proved that the last remainder must measure A and B; consequently the last remainder is the *greatest* common measure of A and B.

Cor. Hence, *if a last remainder can never be arrived at, that is, if the above process be interminable, the proposed magnitudes cannot have a common measure—in other words, they are incommensurable.*

PROPOSITION XIX. THEOREM.

If one magnitude contain another and leave a remainder, such that the greater of the two magnitudes is to the smaller as the smaller to this remainder, then the two magnitudes will be incommensurable.

Let A, the greater of two magnitudes, contain the smaller, B, any number of times, leaving for a remainder a magnitude C, such that A : B : : B : C; then A and B cannot have a common measure.

For let C, D, E, &c. be the successive remainders in the process for finding the common measure (last prop.). Then C cannot measure B, otherwise B would measure A (Prop. X.), and there could be no remainder; but C is contained as often in B as B is contained in A (Def. 7.). Let then P be the greatest multiple of B which is contained in A, and let Q be an equimultiple of C, which must be the greatest contained in B; then (Prop. VI. Cor.) A : P :: B : Q, and (Prop. X.) A : B :: A − P : B − Q; but A − P = C, and B − Q = D, by hypothesis; therefore A : B :: C : D, or B : C :: C : D; hence D cannot measure C, for if it could, C would measure B. Let now P' be the greatest multiple of C in B, and Q' the like multiple of D; then, pursuing the same course as before, there results the proportion C : D :: D : E, so that E cannot measure D; and so on for each succeeding remainder. It appears therefore that no remainder can ever measure the preceding one, consequently the process for finding the common measure of A and B will be interminable, and therefore (Prop. XVIII. Cor.) these two magnitudes are incommensurable.

BOOK VI.

DEFINITIONS.

1. *Similar figures* are such as have the angles of the one respectively equal to those of the other, and the sides containing the equal angles proportional.

2. The *homologous* sides of two similar figures are those, which are each interjacent to two angles respectively equal.

In different circles *similar arcs*, *sectors*, and *segments*, are those of which the arcs subtend equal angles at the centre.

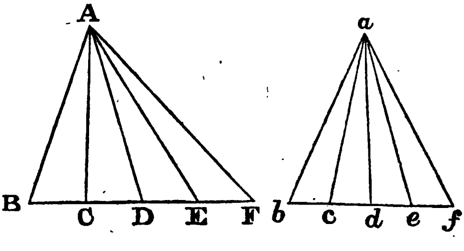
3. If two sides of another figure form the extremes, and two sides of another figure form the means, of a proportion, these sides are said to be *reciprocally proportional.*

Note.—In addition to the symbols employed in the preceding book, we shall, for the sake of brevity, occasionally express the square of a line, AB, by AB^2; and for the rectangle of AB, BC, we shall often write simply $AB \cdot BC$.

PROPOSITION I. THEOREM.

Triangles, whose altitudes are equal, are to each other as their bases.

Let the triangles ABC, *abc*, have equal altitudes, then ABC : *abc* : : BC : *bc*.



For upon the prolongation of the base BC, take any number of distances CD, DE, EF, &c., each equal to BC, and draw AD, AE, AF, &c. In like manner take any number of distances, *ed*, *de*, *ef*, &c. upon the prolongation of *bc*, each equal to *bc*, and draw *ad*, *ae*, *af*, &c.

Then the triangles ABC, ACD, ADE, AEF are equivalent, for they have the same altitude and equal bases (Prop. III. Cor. 2. B. II.); therefore whatever multiple BF is of BC, the same multiple the triangle ABF is of the triangle ABC. For similar reasons whatever multiple *bf* is of *bc*, the same multiple the triangle *abf* is of the triangle *abc*; and if the base BF be greater than the base *bf*, the triangle ABF must be greater than the triangle *abf*, or if the base *bf* be greater than the base BF, the triangle *abf* must be greater than the triangle ABF, for by hypothesis these triangles have equal altitudes. Now the base BF and the triangle ABF are any equimultiples of the base BC and the triangle ABC; also the base *bf* and the triangle *abf* are any equimultiples of the base *bc* and the triangle *abc*; therefore the four magnitudes BC, *bc*, ABC, *abc*, are proportional, for it has been shown that it is impossible to find any equimultiples of the antecedents, and any equimultiples of the consequents, such, that the multiple of one antecedent may be greater than that of its consequent, while the multiple of the other antecedent is not greater than its consequent (Prop. IV. B. V.).

Cor. Hence, *rhomboids whose altitudes are equal, are to each other as their bases*, for rhomboids are the doubles of triangles of the same base and altitude.

Scholium.

The converse of this proposition is obviously true, that is, *triangles which are to each other as their bases have equal altitudes*, for the base of one triangle is to the base of the other, as the former triangle to one of equal altitude upon the latter base, so that if the altitudes were unequal, the triangles could not be to each other as their bases.

PROPOSITION II. THEOREM.

Triangles, whose bases are equal, are to each other as their altitudes.

For every triangle is equivalent to a right angled triangle of equal base and altitude, and in right angled triangles, either of the perpendicular sides being considered as base, the other will be the altitude; therefore, if in two such triangles either the bases or the altitudes be equal, they will, by last proposition, be to each other as the remaining sides; therefore triangles, whose bases are equal, are to each other as their altitudes.

Cor. Therefore, *rhomboids, whose bases are equal, are to each other as their altitudes.*

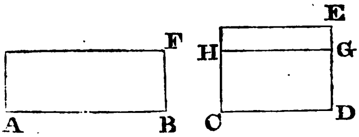
Scholium.

The converse of this proposition, viz. *triangles which are to each other as their altitudes have equal bases,* is evidently true (see preceding scholium).

PROPOSITION III. THEOREM.

If four straight lines are proportional, the rectangle contained by the extremes is equivalent to the rectangle contained by the means, and *conversely,* if two rectangles are equivalent, their containing sides are proportional.

Let the four lines AB, CD, DE, BF be proportional, then AF, the rectangle of the extremes, is equivalent to CE, the rectangle of the means.



Make DG equal to BF, and draw GH parallel to DC.

Then (Prop. I. Cor.) AF : CG : : AB : CD; but by hypothesis, AB : CD : : DE : BF : therefore (Prop. II. B. V.) AF : CG : : DE : BF, or DG, but DE : DG : : CE : CG; therefore (Prop. II. B. V.) AF : CG : : CE : CG, and the consequents being equal the antecedents are equal (Prop. IX. Cor. 2. B. V.), that is, AF ≐ CG.

Conversely. Let the rectangle AF be equivalent to the rectangle CE.

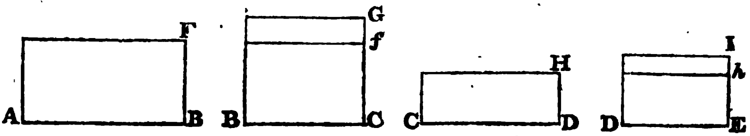
Then (Prop. I. Cor.) AB : CD : : AF or CE : CG, but CE : CG : : DE : DG (Prop. I. Cor.); therefore (Prop. II. B. V.) AB : CD : : DE : DG or BF.

Cor. It follows that *if three lines are in continued proportion, the rectangle of the extremes is equivalent to the square of the mean,* and conversely, *if a square be equivalent to a rectangle, the side of the square is a mean between the sides of the rectangle.*

PROPOSITION IV. THEOREM.

The rectangles contained by the corresponding lines which form two proportions are themselves proportional.

Let there be the proportions AB : BC : : CD : DE, and BF : CG : : DH : EI, then also AF : BG : : CH : DI.



In CG or its production take C*f*=BF, and in EI or its production take E*h*=DH, and through *f* and *h* draw parallels to BC and DE.

Then (Prop. I. Cor. B. VI.) AF : B*f* :: AB : BC :: CD : DE :: CH : D*h*, and alternately, AF : CH :: B*f* : D*h*. Now B*f* : BG :: C*f* : CG :: E*h* : EI :: D*h* : DI; hence, alternately B*f* : D*h* :: BG : DI; but it has just been shown that B*f* : D*h* :: AF : CH; therefore AF : CH :: BG : DI, or AF : BG :: CH : DI.

Cor. 1. Hence *the squares of four proportional lines are proportional.*

Cor. 2. *If three lines are in continued proportion, the square on the first is to the square on the second, as the first line is to the third.* Thus if A, B, C are three lines in continued proportion, then A : B :: B : C, and since A : B :: A : B, we have by the proposition, the square on A to the square on B, as the rectangle of A, B to the rectangle of B, C, and these rectangles are as A to C (Prop. I. Cor.).

Scholium.

The converse of this proposition is not true, for it cannot be inferred that proportional rectangles have proportional sides, since a rectangle may be transformed into an equivalent one, having a side of any given length (Prop. XVII. B. IV.).

The converse of the corollaries are true, that is, first, *if four squares be proportional, their sides will be proportional*; for let A, B, C, D represent the sides of four proportional squares, then if these sides are not proportional let there be the proportion A : B :: C : Q, then, by the corollary, $A^2 : B^2 :: C^2 : Q^2$; but by hypothesis, $A^2 : B^2 :: C^2 : D^2$; consequently (Prop. IX. Cor. 3. B. V.), $Q^2 = D^2$, and therefore Q = D.

Again, *if the squares on two lines are to each other as the first line is to a third, the three lines are in continued proportion,* for let $A^2 : B^2 :: A : C$, and let there be the continued proportion A : B :: B : Q, then, by the corollary, $A^2 : B^2 :: A : Q$; but by hypothesis, $A^2 : B^2 :: A : C$, hence (Prop. IX. Cor. 3. B. V.), Q=C.

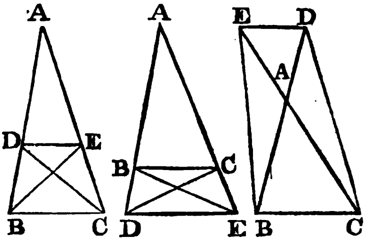
PROPOSITION V. THEOREM.

If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or those produced, proportionally; and *conversely*, if the sides or the sides produced, be cut proportionally, the cutting line will be parallel to the third side of the triangle.

In the triangle ABC let DE be drawn parallel to BC, then AD : DB :: AE : EC.

For join BE, CD.

Then the triangles BDE, DEC are equivalent, for their bases are the same, and being between parallels their altitudes are equal; therefore ADE : BDE :: ADE : DEC, but (Prop. I.), ADE : BDE :: AD : DB, and ADE : DEC :: AE : EC; hence (Prop. II. B. V.) AD : DB :: AE : EC.



Conversely. Let now DE cut the sides AB, AC or their production, so that AD : DB :: AE : EC, then DE will be parallel to BC.

For the same construction remaining, AD : DB :: AED : DEB, and AE : EC :: AED : DEC, hence AED : DEB :: AED : DEC; consequently (Prop. IX. Cor. 2. B. V.), the triangles DEB, DEC are equivalent, and having the same base DE, their altitudes are equal, that is, they are between the same parallels.

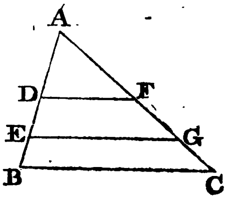
Cor. AD+DB : AD :: AE+EC : AE, that is, AB : AD :: AC : AE, also AB : BD :: AC : CE.

PROPOSITION VI. THEOREM.

If through two sides of a triangle two lines be drawn parallel to the third side, the portions intercepted will be to each other as the sides themselves.

Let the two sides AB, AC of the triangle ABC be divided by the lines DF, EG parallel to BC; then AB : AC :: DE : FG.

For by last proposition AD : AF :: DE : FG, and by the corollary AD : AF :: AB : AC; consequently, AB : AC :: DE : FG.



Cor. Hence *if any number of parallels be drawn, the sides will be cut proportionally, the opposite intercepted portions being to each other as the sides themselves.*

Scholium.

The converse of this proposition does not follow, that is, it is not true, that if the portions intercepted by two lines cutting two sides of a triangle are to each other as those sides, the cutting lines will be parallel to the third side of the triangle; for any two lines drawn from D, E to intercept on the other side a portion equal to FG, will be drawn as this converse directs, although only a single pair can be parallel to the side BC.

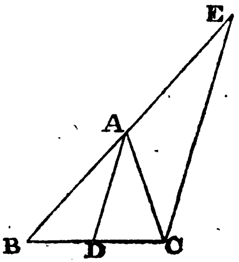
PROPOSITION VII. THEOREM.

The line which bisects any angle of a triangle divides the opposite side into portions, which are to each other as the adjacent sides.

Let AD bisect the angle A of the triangle ABC, then BD : DC :: AB : AC.

Draw CE parallel to DA, meeting BA produced in E.

Then (Prop. V.) BD : DC :: BA : AE. Now because AD, EC are parallel, the angle E is equal to the angle BAD, and the angle ACE to the angle CAD, which is equal, by hypothesis, to BAD; it appears then that the angles E and ACE are each equal to BAD; therefore AE is equal to AC; hence, putting AC for AE in the above proportion, we have BD : DC :: BA : AC.



PROPOSITION VIII. THEOREM. (*Converse of Prop. VII.*)

If a line from the vertex of any angle of a triangle divide the side opposite into portions, which are to each other as the sides adjacent, the line so drawn bisects the angle.

In the triangle ABC (preceding diagram) let the line AD divide BC, so that BD : DC :: BA : AC, then is the angle BAD equal to the angle CAD.

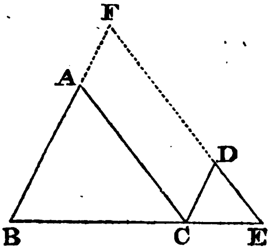
Draw CE parallel to DA, meeting BA, produced in E.

Then (Prop. V.) BD : DC :: BA : AE; but by hypothesis, BD : DC :: BA : AC, therefore BA : AE :: BA : AC; consequently, AE = AC, and therefore the angle ACE to the angle AEC; but because of the parallels AD, EC, the angles ACE, AEC are respectively equal to the angles DAC, DAB; these angles are therefore equal, and consequently AD bisects the angle BAC.

PROPOSITION IX. THEOREM.

If two triangles have the angles of the one respectively equal to those of the other, the sides containing the equal angles are proportional.

Let the triangles ABC, DCE have the angles A, B, in the one, respectively equal to the angles D, C, in the other; these triangles will be similar.



For let them be placed so that two homologous sides BC, CE may form one straight line, and produce BA, ED till they meet in F.

Then since the angle B is equal to the angle DCE, the line BF is parallel to the line CD; also since the angle ACB is equal to the angle E, the line AC is parallel to the line FE; therefore CF is a rhomboid, and consequently AF = CD, and FD = AC. Now because AC is parallel to FE, we have (Prop. V.) BC : CE :: BA : AF, and because CD is parallel to BF we have BC : CE :: FD : DE, and if in these proportions CD be put for its equal AF, and AC for its equal FD, they become

BC : CE :: BA : CD
BC : CE :: AC : DE

whence BA : CD :: AC : DE.

Therefore the sides containing the equal angles are proportional.

Cor. 1. Hence (Def. 1.), *if the angles of one triangle are respectively equal to those of another, the triangles are similar.*

Cor. 2. Therefore *if a line be drawn parallel to one of the sides of a triangle it will cut off a triangle similar to the whole.*

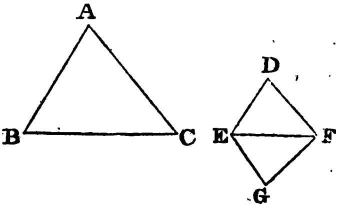
PROPOSITION X. THEOREM. (*Converse of Prop. IX.*)

The angles of one triangle are respectively equal to those of another, if the containing sides are proportional.

In the triangles ABC, DEF, let there be the following proportions among the sides, viz.

AB : BC : : DE : EF
AC : CB : : DF : FE;

then will the angles B, C, A be respectively equal to the angles E, F, D.



For, let EG, FG be drawn, making angles at E and F, with the side EF respectively equal to the angles B and C; then the triangles ABC, GEF have the angles in the one respectively equal to those of the other, and, consequently (Prop. IX.), the containing sides are proportional; so that

AB : BC : : EG : EF;
but AB : BC : : DE : EF, by hypothesis: hence (Prop. IX. Cor. 3. B. V.) EG is to equal to DE.

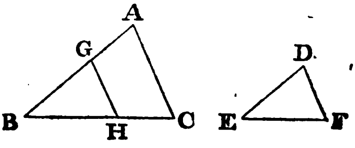
Again, AC : CB : : FG : FE;
but AC : CB : : DF : FE, by hypothesis; therefore, FG is equal to DF. Since, then, the sides of the triangle EGF are respectively equal to those of the triangle DEF, these triangles are equal; but the angles of the triangle EGF are respectively equal to those of the triangle ABC; therefore the angles of the triangle DEF are respectively equal to those of the triangle ABC.

Cor. Hence, *triangles whose corresponding sides are proportional, are similar.* Indeed the above demonstration establishes the similarity of the triangles, if they have the sides containing *two* angles in each proportional.

PROPOSITION XI. THEOREM.

Triangles are similar, which have an angle in the one equal to an angle in the other, and the containing sides proportional.

Let the triangles ABC, DEF have the angle B in the one equal to the angle E in the other, while the containing sides form the proportion AB : BC : : DE : EF; the triangles are similar.



For, make BG, BH respectively equal to ED, EF, and join GH.

Then the triangles GBH, DEF are equal, since two sides and the included angle in the one are respectively equal to two sides and the included angle in the other; hence, and by hypothesis, AB : BC : : GB : BH, that is, the sides BA, BC of the triangle ABC are cut proportionally by the line GH; GH, therefore, is parallel to AC (Prop. V.): hence (Prop. IX. Cor. 2.) the triangle GBH is similar to the triangle ABC, and the triangle DEF has been shown to be equal to the triangle GBH; therefore the triangle DEF is similar to the triangle ABC.

Scholium.

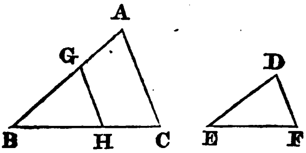
The above proposition is obviously true of rhomboids, that is, ***rhomboids are similar which have an angle in the one equal to an angle in the other, and the containing sides proportional;*** for such rhomboids must be equiangular, and the opposite sides of each being equal, it follows that the sides containing the equal angles are proportional; therefore (Def. 1.) they are similar.

PROPOSITION XII. THEOREM.

Triangles are similar which have an angle in each equal, and the sides containing another angle proportional, provided the third angle in each be of the same character.

In the triangles ABC, DEF let the angles B, E be equal, while the angles A, D are both either right, obtuse, or acute;

and let there also be the proportion BC : CA : : EF : FD, then the triangles are similar.



If BC is equal to EF, then, by the proportion, CA is equal to FD; so that in this case the triangles, having two sides and an opposite angle in each respectively equal, while the other opposite angles have the same character, are equal (Prop. XXVI. B. I.), and, therefore, necessarily similar. But, if one of these sides as BC is greater than the other EF, let BH be equal to EF, and draw HG parallel to CA; then the triangles ABC, GBH are similar (Prop. IX. Cor. 2.); so that

BC : CA : : BH : HG; but, by hypothesis,
BC : CA : : EF : FD,

and, by construction, BH$=$EF, therefore (Prop. IX. Cor. 3. B. V.) HG$=$FD. The two triangles GBH, DEF, having thus two sides, and a corresponding opposite angle in each respectively equal, while the other opposite angles G, D have the same character, are equal; but the triangle GBH is similar to the triangle ABC; therefore the triangle DEF is also similar to the triangle ABC.

Cor. If the equal angles in each triangle be either obtuse or right, then, since the other angles must be acute, it follows that *triangles are similar which have either a right or an obtuse angle in each equal, and the sides containing another angle in each proportional.*

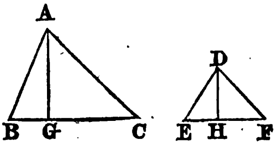
Scholium.

The converse of the two last propositions is obviously included in the definition of similar figures.

PROPOSITION XIII. THEOREM.

In similar triangles the bases are as the altitudes.

Let ABC, DEF be similar triangles, the base BC is to the base EF as the altitude AG to the altitude DH.



For, since the angles B, C are by hypothesis equal, the right angled triangles ABG, DEH are similar (Prop. IX.); therefore BG : EH : : AG : DH. For similar

reasons GC : HF :: AG : DH; consequently BG : EH :: GC : HF, or BG : GC :: EH : HF; therefore (Prop. XI. Cor. 2. B. V.) BG+GC : EH+HF :: BG : EH, but BG : EH :: AG : DH; hence

BC : EF :: AG : DH.

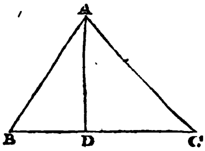
Scholium.

The converse of this proposition fails, for all triangles of the same base and altitude being equivalent, it is obvious that two may have their bases as their altitudes without being similar.

PROPOSITION XIV. THEOREM.

The perpendicular, from the vertex of the right angle to the hypothenuse of a right angled triangle, divides it into two triangles similar thereto.

In the triangle ABC right angled at A, let the perpendicular AD be drawn, then will each of the triangles ABD, ACD be similar to the triangle ABC, and, consequently, similar to each other.



For the triangles ABD, ABC have the angle B in common, and they are both right angled; they are, therefore, similar (Prop. IX.). For like reasons the triangles ACD, ACB are also similar; each of the triangles ABD, ACD are, therefore, similar to the triangle ABC.

Cor. 1. Hence BC : BA :: BA : BD, that is, *a side of a right angled triangle is a mean proportional between the hypothenuse and that part of it intercepted by the proposed side, and the perpendicular from the vertex of the right angle.*

Cor. 2. The similar triangles ABD, ACD furnish the proportion BD : AD :: AD : DC, that is, *the perpendicular from the vertex of the right angle to the hypothenuse, is a mean between the parts into which the hypothenuse is divided by it.*

Scholium.

By combining the relations exhibited in these corollaries, we arrive very easily at the celebrated property of the right angled triangle, which forms Proposition X. of Book I.; for, from Cor. 1. and Prop. III. it appears that $AB^2 = BD \cdot BC$, and $AC^2 = DC \cdot BC$; therefore by addition $AB^2 + AC^2 = BD \cdot BC +$

$DC \cdot BC = (BD+DC) \cdot BC$ (Prop. IV. B. II.); but $(BD+DC) = BC$; therefore $AB^2+AC^2=BC^2$, which is the property referred to.

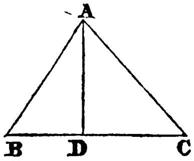
We may here add the remark which Legendre has subjoined to this proposition, viz. That it frequently happens, as in this instance, that by deducing consequences from one or more propositions, we are led back to some proposition already proved. In fact, the chief characteristic of geometrical theorems, and one indubitable proof of their certainty is, that however we combine them together, provided only our reasoning be correct, the results we obtain are always perfectly accurate. The case would be different if any proposition were false or only approximately true; it would frequently happen that, on combining the propositions together, the error would increase and become perceptible. Examples of this are to be seen in all the demonstrations, in which the *reductio ad absurdum* is employed. In such demonstrations, where the object is to show that two magnitudes are equal, we proceed by showing that if there existed the smallest inequality between them, a train of accurate reasoning would lead us to a manifest absurdity; from which we are forced to conclude that the two magnitudes are equal.

PROPOSITION XV. THEOREM. (*Converse of Prop. XIV.*)

If a triangle is divided into two similar triangles, by a perpendicular from the vertex of one of its angles to the opposite side, that angle shall be a right angle.

Let the perpendicular AD divide the triangle ABC into the two similar triangles ABD, ACD, then the angle BAC is a right angle.

For one of the angles of this triangle must necessarily be right, otherwise it could not be similar to the right angled triangles into which it is divided: now, neither B nor C can be right, since neither are parallel to AD; hence BAC must be a right angle.



Scholium.

It is further obvious that, *if from the vertex of the right angle a line drawn to the hypothenuse divide the triangle into two triangles similar thereto, the line so drawn will be perpendicular to the hypothenuse*; for neither B nor C can be a right angle, hence ADB must.

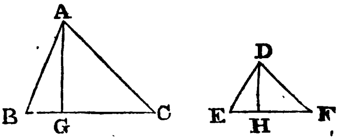
It appears, likewise, that the right angled triangle is the only triangle that can be divided into two triangles similar to it; for, in all cases, the exterior angle ADB is greater than either of the angles ACD, DAC; and, therefore, if the triangles ABD, ACD are similar, the angle ADB must be equal to the angle ADC, AD must, therefore, be perpendicular to BC, and, by the proposition, the angle BAC must in consequence be right. It follows, moreover, that no triangle can be divided into two scalene triangles that shall be similar to each other.

PROPOSITION XVI. THEOREM.

Triangles having an angle in the one equal to an angle in the other, are to each other as the rectangles of their containing sides.

Let the triangles ABC, DEF have the angle B in the one equal to the angle E in the other, then

$$ABC : DEF :: AB \cdot BC : DE \cdot EF.$$



For, draw AG perpendicular to BC, and DH perpendicular to EF.

Then the triangles ABG, DEH are similar, consequently $AB : AG :: DE : DH$; now

(Prop. I. Cor.) $AB \cdot BC : AG \cdot BC :: AB : AG$,
and $DE \cdot EF : DH \cdot EF :: DE : DH$;

therefore $AB \cdot BC : AG \cdot BC :: DE \cdot EF :: DH \cdot EF$; now (Prop. III. Cor. V. B. II.) $AG \cdot BC$ is double the triangle ABC, and $DH \cdot EF$ is double the triangle DEF; therefore, (Prop. X. Cor. 1. B. V.) $AB \cdot BC : DE \cdot EF :: ABC : DEF$.

Cor. 1. Hence, *if the rectangles of the sides containing the equal angles be equivalent, the triangles will be equivalent, and if triangles having an angle in the one equal to an angle in the other be equivalent, the rectangles of the sides containing the equal angles will be equivalent; the same is also true of rhomboids having an angle in each equal, for each rhomboid is composed of two such triangles* (Prop. II. Cor. 1. B. II.).

Cor. 2. Therefore, *triangles having an angle in the one equal to an angle in the other, and the containing sides reciprocally proportional, are equivalent;* and, conversely, *equivalent triangles having an angle in each equal, have their con-*

taining sides reciprocally proportional, and the same is true of rhomboids.

Cor. 3. Since the angles B, E are equal, the right angled triangles ABG, DEH are similar, and furnish the proportion

AB : DE : : BG : EH,

and BC : EF : : BC : EF;

therefore (Prop. IV.) AB·BC : DE·EF :: BG·BC : EH·EF;

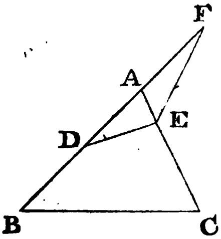
consequently *the triangle ABC is to the triangle DEF as the rectangle BG·BC is to the rectangle EH·EF; and if BG= BC, that is, if C be a right angle, then the triangles are to each other as* BC^2 *is to EH·EF.*

Scholium.

The converse of the above proposition is not true, viz. if two triangles are to each other as the rectangles of two sides in each, these sides will include equal angles.

Let ABC, ADE be two triangles, having a common angle A. Produce DA till AF is equal to it, and join FE.

Then ABC : ADE : : AB·AC : AD·AE, and since AF= AD, the triangles AEF, AED are equivalent; therefore



ABC : AEF : : AB·AC : AF·AE;

now the angles BAC, EAF are equal only when BAC is a right angle; in every other case therefore this converse fails.

This circumstance affords us an opportunity of exemplifying to the student one of the advantages arising from attending to converse propositions. It is doubtless both interesting and important to know, when any particular property has been shown to belong to figures constructed under certain conditions, whether this property be peculiar to those figures, or whether it be not at least possible for some other figure of the same class, but under a different modification of form, to possess the same property. Converse propositions enable to discover this:—If the converse is true, we infer that whenever the property or relation specified in the direct proposition is found to exist in any figure coming under the same general definition, it is necessarily one of those which the proposition distinguishes. But if the converse fail, we then conclude that the property demonstrated to exist under the proposed conditions, exists also under different conditions, and this must necessarily happen when the proposition is only a particular case of one more general; and, therefore, it is advisable when these failures occur to examine whether the proposition cannot be rendered more comprehensive. The present proposition is of this kind, being included in the following:—

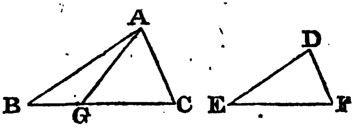
Triangles having an angle in the one equal to an angle in the other, or having an angle in the one, together with an angle in the other equal to two right angles, are to each other as the rectangles of the containing sides. To show this it is only necessary to remark, that it is proved in the scholium to Prop. XXIX. Book I., and may also be inferred from Cor. 2. to Prop. III. Book II., that two triangles are equivalent, when two sides of the one are respectively equal to two sides of the other, and the sum of the included angles equal to two right angles; and, consequently, the proportion $AB \cdot BC : DE \cdot EF :: ABC : DEF$, to which the above demonstration conducted will continue true, although we substitute for one of the triangles, as ABC, another, having the same sides AB, BC, but including an angle, making with the angle E two right angles. The converse of this proposition will be:—*If two triangles are to each other as the rectangles of two sides in each, these sides will include either equal angles, or else angles whose sum will make two right angles*, which the student will not find much difficulty in demonstrating.

PROPOSITION XVII. THEOREM.

Similar triangles are to each other as the squares of their homologous sides.

Let the triangles ABC, DEF be similar, and let BC, EF be homologous sides; that is, let the angles B, C be respectively equal to the angles E, F, then

$$ABC : DEF :: BC^2 : EF^2.$$



For, let BG be a third proportional to BC, EF; that is, let there be the proportion BC : EF :: EF : BG.

Because the triangles are similar, AB : BC :: DE : EF, and alternately AB : DE :: BC : EF; therefore AB : DE :: EF : BG; that is, the sides about the equal angles B, E of the triangles ABG, DEF are reciprocally proportional; therefore (Prop. XVI. Cor. 2.) these triangles are equivalent. Now, BC : BG : : ABC : ABG, but (Prop. IV. Cor. 2.) BC : BG : : BC^2 : EF^2; therefore, putting the triangle DEF for its equal ABG, we have

$$ABC : DEF :: BC^2 : EF^2,$$

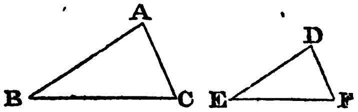
or, since the squares of proportional lines are proportional,

$$ABC : DEF :: BC^2 : EF^2 :: AB^2 : DE^2 :: AC^2 : DF^2.$$

Cor. Since triangles of the same altitude are to each other as their bases, it follows (Prop. XIV.) that *in a right angled triangle the squares of the sides are to each other as the adjacent parts into which the perpendicular from the vertex of the right angle divides the hypothenuse*, that is, in the diagram to proposition XIV. $AB^2 : AC^2 :: BD : DC$.

PROPOSITION XVIII. (*Converse of Prop. XVII.*)

Triangles which are to each other as the squares of their respective sides are similar.



Let the triangles ABC DEF be to each other as the squares of their respective sides, that is,

$$ABC : DEF :: BC^2 : EF^2 :: AB^2 : DE^2 :: AC^2 : DF^2.$$

Then, since the sides of proportional squares are proportional,

$$BC : EF :: AB : DE :: AC : DF;$$

hence, the angles of the triangle ABC are respectively equal to those of the triangle DEF (Prop. X.), that is, the triangles are similar.

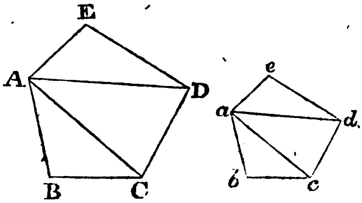
PROPOSITION XIX. THEOREM.

Similar polygons may be divided into the same number of triangles, similar each to each, and similarly situated; and, *conversely*, polygons which are composed of the same number of triangles, similar each to each, and similarly situated, are themselves similar.

Let the polygons in the margin be similar, the angle A being equal to the angle *a*, the angle B to the angle *b*, and so on;

while the sides containing these angles are proportional; they may be divided into the same number of similarly situated triangles, which will be similar each to each.

For, from the vertices of two corresponding angles as A, *a*, let diagonals be drawn.



Then, because the polygons are similar, AB : BC : : *ab* : *bc*, from which we infer the similarity of the triangles ABC, *abc* (Prop. XI.), and they evidently occupy similar situations in the two polygons. The angles ACB, *acb* are, therefore, equal, and since the angles BCD, *bcd* are also equal, the angles ACD, *acd* are equal. Now the similar polygons furnish the proportion

BC : CD : : *bc* : *cd*, and the similar triangles ABC,
abc give BC : AC : : *bc* : *ac*;
hence AC : CD : : *ac* : *cd*;

therefore (Prop. XI.) the triangles ACD, *acd* are similar, and they are similarly situated. In like manner may the triangles ADE, *ade* be proved to be similar, and so on in every pair of corresponding triangles; hence, similar polygons may be divided into the same number of triangles, similar each to each, and similarly situated.

Conversely. Let the polygons be composed of the same number of similar triangles similarly situated, the polygons are similar.

For, since the corresponding triangles are similar, the following angles are equal each to each, viz. B=*b*, BCA=*bca*, ACD=*acd*, CDA=*cda*, ADE=*ade*, and so on round the two polygons; therefore, by addition, we shall find the angles of the one polygon equal to the corresponding angles in the other, that is, B=*b*, BCD=*bcd*, CDE=*cde*, and so on; we have, moreover, the proportions

AB : *ab* : : BC : *bc* : : AC : *ac* : : CD : *cd*, &c.

therefore the polygons have their angles respectively equal, and the containing sides proportional.

PROPOSITION XX. THEOREM.

The perimeters of similar polygons are to each other as their homologous sides.

Let the polygons in the opposite margin be similar, they will be to each other as their homologous sides.

For, since the polygons are similar, we have the following proportions among their homologous sides, viz.

$$AB : ab :: BC : bc :: CD : cd, \text{ \&c.};$$

and, since in a series of proportionals, one antecedent is to its consequent as the sum of the antecedents to the sum of the consequents (Prop. VI. B. V.), we have

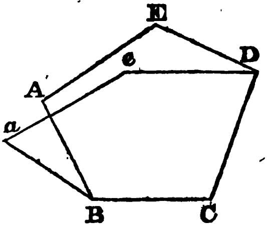
$$AB : ab :: ABCD \text{ \&c.} : abcd \text{ \&c.};$$

that is, the perimeters are as their homologous sides.

Cor. The perimeters are also as the corresponding diagonals.

Scholium.

The converse of this proposition is not true, viz. if the perimeters of two polygons are as their respective sides, the polygons will be similar.

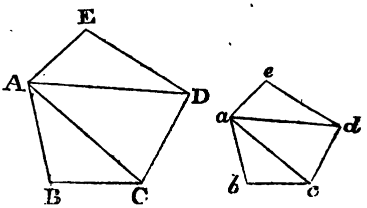


Let ABCDE be any polygon, and from D draw D*e*, making an angle with DC less than the angle EDC, and let D*e* be equal to DE; then, if from *e* as a centre, with a radius equal to EA, an arc be described, and from B as a centre, with a radius equal to BA another be described, cutting the former in *a*, and if the lines *ea*, B*a* be drawn, it is obvious that the sides of the polygon *a*BCD*e* will be respectively equal to those of the polygon ABCDE, although they will be dissimilar, since their angles are not respectively equal. Now, if a polygon similar to either of these be constructed, their perimeters will be as their respective sides, and if the dissimilar polygon be compared with this new polygon, the perimeters will in like manner be as their respective sides; hence the converse of the proposition fails. In triangles, however, the converse holds, for if the respective sides are to each other as their perimeters, these perimeters are to each other as $AB : ab :: BC : bc :: AC : ac$, that is, the homologous sides are proportional. Although, therefore, the proportionality of the sides is sufficient to establish the similarity of triangles, it is not sufficient for the higher polygons; and in like manner, although in triangles equality of angles establishes proportionality of sides; yet, in other polygons, this proportionality is not a consequence of such equality in their corresponding angles, for a parallel to one of the sides of a polygon, passing through the two adjacent sides and cutting off a portion of the polygon, will, with the remaining sides, form a polygon of the same angles, although the proportionality of the sides must necessarily be altered.

PROPOSITION XXI. THEOREM.

In similar polygons the surfaces are to each other as the squares of their homologous sides.

Let the polygons in the margin be similar, and let them be divided into similar triangles, by diagonals drawn from the vertices of the equal angles A, *a*.



Then, by comparing these similar triangles, we have the proportions

$AC^2 : ac^2 :: ABC : abc$

$AC^2 : ac^2 :: ACD : acd$

and since $AC^2 : ac^2 :: AD^2 : ad^2$; we have moreover

$AC^2 : ac^2 :: ADE : ade$,

therefore

$$ABC : abc :: ACD : acd :: ADE : ade,$$

consequently

$$ABC + ACD + ADE : abc + acd + ade :: ABC : abc :: AB^2 : ab^2,$$

that is, the polygons ABCDE, *abcde* are to each other as the squares of their homologous sides.

Cor. 1. *Similar polygons are to each other as the squares of any corresponding diagonals.*

Cor. 2. *If similar polygons have a side in one equal to an homologous side in the other, the polygons must be equal.*

Scholium.

The converse of this proposition fails; viz. If the surfaces of two polygons are to each other as the squares of their corresponding sides, the polygons are similar.

Let the surfaces of the two polygons ABCDE, *abcde* be to each other as

$$AB^2 :: ab^2 : BC^2 :: bc^2 : CD^2 :: cd^2, \&c$$

Now it has been shown in the scholium to the preceding proposition, that the proportionality of the sides is insufficient to establish the similarity of the polygons; but from this proportionality which is thus expressed,

$$AB : ab :: BC : bc :: CD : cd, \&c.$$

the following is derived (Prop. VI. Cor. 1.),

$$AB^2 : ab^2 :: BC^2 : bc^2 :: CD^2 : cd^2, \&c.$$

therefore this is insufficient to establish the similarity of the polygons; consequently the converse fails.

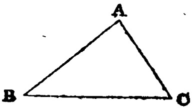
PROPOSITION XXII. THEOREM.

If similar polygons be described upon the two sides, and upon the hypothenuse of a right angled triangle, that on the hypothenuse will be equivalent to both those on the sides; and, *conversely*, if similar polygons be described on the sides of a triangle, and if that on the longest side be equivalent to both the others, the angle opposite to that side will be right.

Let the triangle ABC be right angled at A, then whatever rectilineal figure be described on BC it shall be equivalent to both the similar figures described on AB and AC.

For let us denote the figure on BC by X, that on AC by Y, and that on AB by Z. Then by the preceding proposition,

$$X : BC^2 :: Y : AC^2 :: Z : AB^2;$$



therefore (Prop. VI. B. V.)

$$X : BC^2 :: Y+Z : AC^2+AB^2;$$

but (Prop. X. B. II.) $BC^2 = AC^2 + AB^2$, therefore $X = Y + Z$; hence the polygon on BC is equivalent to the similar polygons on AB, AC.

Conversely. Let X, Y, Z denote the three similar polygons described upon BC, AC, AB, the respective sides of the triangle ABC, then if $X = Y + Z$, the angle A will be a right angle.

For the polygons being similar

$$X : BC^2 :: Y : AC^2 :: Z : AB^2;$$

therefore

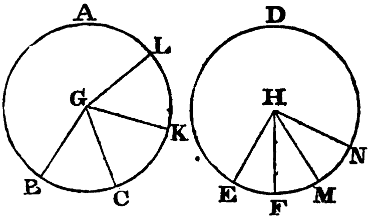
$$X : BC^2 :: Y+Z : AC^2+AB^2.$$

Now by hypothesis, $X=Y+Z$; therefore $BC^2=AC^2+AB^2$, and consequently (Prop. XII. Cor. 1. B. II.) the angle A is right.

PROPOSITION XXIII. THEOREM.

In equal circles, angles at the centres are to each other as the arcs which they subtend.

Let the circles ABC, DEF be equal, and let BGC, EHF be angles at their centres, then the angle BGC is to the angle EHF as the arc BC is to the arc EF.



Upon the circumference ABC take any number of consecutive arcs CK, KL, &c., each equal to BC, and in like manner upon the circumference DEF take any number of arcs, FM, MN, &c. each equal to EF; draw GK, GL; HM, HN.

Then, because the arcs BC, CK, KL are equal, the angles which they subtend at the centre are also equal; therefore whatever multiple the arc BL is of the arc BC, the same multiple is the angle BGL of the angle BGC. For similar reasons whatever multiple the arc EN is of the arc EF, the same multiple is the angle EHN of the angle EHF; and if the arc BL be longer than the arc EN, the angle BGL must be greater than the angle EHN, or if the arc EN be longer than the arc BL, the angle EHN must be greater than the angle BGL. Now the arc BL and the angle BGL are any equimultiples whatever of the arc BC and the angle BGC, also the arc EN and the angle EHN are any equimultiples of the arc EF and the angle EHF; consequently (Prop. IV. B. V.)

angle BGC : *angle* EHF : : *arc* BC : *arc* EF.

Cor. 1. It follows that *an angle at the centre is to four right angles, as the arc which subtends it is to the whole circumference*; for the angle BGC is to CGK as the arc BC to CK, and the angle BGC is to KGL as the arc BC to KL, and so on round the circumference; therefore (Prop. XVI. Schol. B. V.) the angle BGC is to the sum of the angles about G as the arc BC to the sum of the arcs BC, CK, KL, &c. round the circumference.

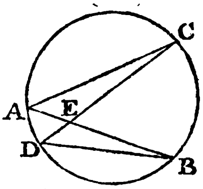
Cor. 2. It is obvious that the sectors BGC, CGK, KGL, &c. are equal, since they would coincide, if applied to each other; in like manner, the sectors EHF, FHM, MHN are also equal; consequently, if in the preceding demonstration, we were to substitute *sectors* for *angles*, it would follow that *in equal circles sectors are to each other as their arcs*, and in the same circle *any sector is to the whole circle as its arc is to the circumference.*

PROPOSITION XXIV. THEOREM.

If two chords intersect each other, the rectangle contained by the parts of the one is equivalent to the rectangle contained by the parts of the other.

Let the chords AB, CD intersect in E, then AE·EB = CE·ED.

For join AC, BD.



Then the angles B, C, subtended by the arc AD, are equal, and the angles BED, CEA being also equal, it follows that the triangles AEC, DEB are similar, therefore AE : DE : : EC : EB; consequently AE·EB = DE·EC.

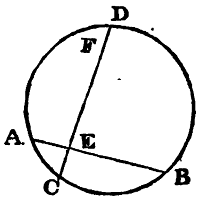
Cor. 1. *The parts of two chords intersecting each other are reciprocally proportional.*

Cor. 2. If one of the chords be a diameter, and if the other be perpendicular to it, then, since it is bisected thereby (Prop. V. B. III.), it follows *that the rectangle of the parts into which a perpendicular chord divides a diameter, is equivalent to the square of half that chord.*

PROPOSITION XXV. THEOREM. (*Converse of Prop. XXIV.*)

If two straight lines intersect each other so that the rectangle of the parts of the one may be equivalent to the rectangle of the parts of the other, a circumference may be described through their extremities.

Let AB, CD intersect in E, so that AE·EB = DE·EC, then the points A, C, B, D, all lie in the same circumference.



For suppose that a circumference is described through three of the points, as A, C, B, (Prop. VIII. B. III.), and that it intersects CD in F.

Then, by last proposition, $AE \cdot EB = FE \cdot EC$; but by hypothesis, $AE \cdot EB = DE \cdot EC$; hence the rectangles $FE \cdot EC$ and $DE \cdot EC$ are equivalent, and one side EC is common to both; therefore the other sides FE, DE are equal (Prop. III. Schol. 2. B. III.), so that the points F and D coincide.

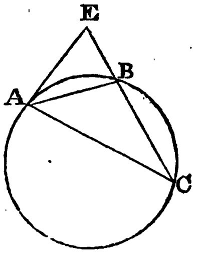
Cor. Hence *if the diagonals of a quadrilateral intersect each other, so that the rectangle of the parts of the one may be equivalent to the rectangle of the parts of the other, a circumference may be circumscribed about it.*

PROPOSITION XXVI. THEOREM.

If from a point without a circumference two straight lines be drawn, one to touch the circumference, and the other to cut it in two points, the square of the tangent will be equivalent to the rectangle contained by the other line, and that part of it which is without the circle.

From the point E let the lines EA, EBC be drawn, the former touching the circumference ACB, and the latter cutting it, then $EA^2 = EC \cdot EB$.

Draw AB, AC.



Then the triangles EAC, EBA having the angle E in common, and EAB, included by a tangent and a chord, equal to the angle C in the alternate segment (Prop. XV. B. III.), it follows that they are similar, so that $EC : EA :: EA : EB$; therefore

$$EA^2 = EC \cdot ED.$$

Cor. 1. Hence, *if from a point one line be drawn to touch, and another to cut, the circumference, the former will be a mean proportional between the latter and that part of it which is without the circle.*

Cor. 2. Also, *if from the same point two lines be drawn to cut the circumference, the rectangle contained by the whole and the external part of the one will be equivalent to the rectangle contained by the whole and the external part of the other*, each rectangle being equivalent to the square of the tangent from the same point.

Cor. 3. Consequently, *of two lines so drawn the wholes will be reciprocally proportional to their external parts.*

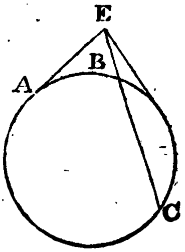
Cor. 4. *Two tangents drawn from the same point are equal.*

Cor. 5. And since a radius drawn to the point of contact is perpendicular to the tangent, it follows that *the angle included by two tangents drawn from the same point is bisected by a line drawn from the centre of the circle to that point*, for this line forms the hypothenuse common to two equal right angled triangles.

PROPOSITION XXVII. *(Converse of Prop. XXVI.)*

If from a point without a circumference, two lines be drawn to it, of which one cuts it, so that the rectangle of the whole line and the external part may be equivalent to the square of the other line, this latter is a tangent to the circle.

Let the lines EC, EA be drawn, the former cutting the circumference ABC in the points B, C, so that $EC \cdot EB = EA^2$, then EA touches the circumference.



For if EA be supposed to cut the circumference, let two tangents be drawn from E, then (last Prop. Cor. 1.) each is a mean proportional between EC, EB; but by hypothesis, EA is also a mean proportional between EC, EB; hence EA is equal to each of the tangents, that is, from the same point, not the centre, three equal lines are drawn to the circumference, which is impossible (Prop. VIII. Cor. 2. B. III.); hence EA cannot cut the circumference.

Scholium.

1. The converse of corollary 1. to proposition XXII. being

nothing more than the proposition itself, expressed in different terms, is therefore true.

2. With respect to the second corollary the converse is also true, viz. If from the same point two lines be drawn and be divided, so that the rectangle of the whole, and the part intercepted between their concourse and the point of division, may be the same in each, a circumference may be described through the other extremities of the lines and the points of division. This may be proved as the converse of Prop. XXIV. by showing that the circumference passing through three of the points must necessarily pass also through the fourth (see prop. XXV.)

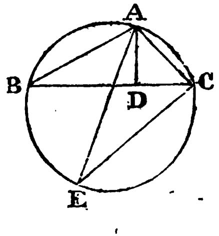
3. The third corollary is the second differently expressed, its converse is therefore true. The converse of the remaining corollaries do not obtain.

PROPOSITION XXVIII. THEOREM.

In every triangle the rectangle of two sides is equivalent to the rectangle contained by the perpendicular, from the vertex of their included angle to the third side, and the diameter of the circumscribing circle.

Let ABC be a triangle circumscribed by the circle ABEC, of which AE is the diameter, and let AD be perpendicular to BC, then AB·AC=AD·AE.

Join CE.



Then the triangles ABD, AEC are right angled at D and C (Prop. XIV. Cor. 3. B. III.), and the angles B and E, subtended by the arc AC, are equal; these triangles are therefore similar, so that AB : AD :: AE : AC, consequently,

$$AB\cdot AC = AD\cdot AE.$$

Scholium.

1. If the angle A were right, then BC would be equal to AE (Prop. XV. Cor. 2. B. III.), so that the above proportion would be AB : AD :: BC : AC, consequently AB : BC ::

$AD : AC$; and from these proportions we infer (Prop. XII.) that the perpendicular from the vertex of the right angle to the hypothenuse divides the triangle into two similar ones, so that proposition XIV. is immediately deducible from the above proposition, which may be otherwise enunciated, thus:—Any side of a triangle is to the perpendicular from its extremity to another side, as the diameter of the circumscribing circle is to the third side.

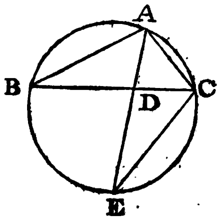
2. It is plain, that the converse of the proposition is true, that is, if the rectangle of two sides of a triangle is equivalent to a rectangle, of which one side is the perpendicular upon the third side of the triangle, the other side of the rectangle will be equal to the diameter of the circumscribing circle; or if one side of the rectangle be equal to the diameter of the circumscribing circle, the other side will be equal to the perpendicular.

PROPOSITION XXIX. THEOREM.

In every triangle the rectangle of two sides is equivalent to the rectangle of the parts into which the line, bisecting their included angle, divides the third side, together with the square of this line.

If in the triangle ABC the line AD bisect the angle A, then $AB{\cdot}AC = BD{\cdot}DC + AD^2$.

For let a circumference be described through the points A, B, C, and let AD be prolonged till it meets it in E, and join E, C.



Then the triangles BAD, EAC have, by hypothesis, the angles BAD, EAC equal, while the angles B, E are in the same segment; these triangles therefore are similar; hence $AB : AE :: AD : AC$, consequently, $AB{\cdot}AC = AE{\cdot}AD$; but (Prop. IV. B. III.) $AE{\cdot}AD = AD{\cdot}DE + AD^2$, and (Prop. XXIV.) $AD{\cdot}DE = BD{\cdot}DC$, therefore

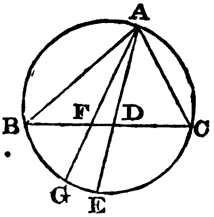
$$AB{\cdot}AC = BD{\cdot}DC + AD^2.$$

PROPOSITION XXX. (*Converse of Prop. XXIX.*)

If a line drawn from the vertex of any angle of a triangle divide the opposite side, so that the rectangle of the parts, together with the square of the dividing line, may be equivalent to the rectangle of the other two sides of the triangle, that line will bisect the angle, from whose vertex it is drawn, *except it be the vertical angle of an isosceles triangle.*

Let ABC be a triangle, and suppose AD to be drawn, so that $BD \cdot DC + AD^2 = AB \cdot AC$, then AD bisects the angle BAC.

For let the triangle be circumscribed by a circle, and let the prolongation of AD meet the circumference in E, and if the angle BAC is not bisected by this line, let some other, as AFG, bisect it, and join GE.



Then, by last proposition, $AB \cdot AC = AG \cdot AF$; but by hypothesis, $AB \cdot AC = AE \cdot AD$, for $BD \cdot DC + AD^2 = AE \cdot AD$, consequently $AG \cdot AF = AE \cdot AD$, so that if a circumference were to be described through the points E, D, F, it would also pass through the point G (Prop. XXVII. Schol. 2.); therefore the opposite angles F, E, of the quadrilateral FGED, are together equal to two right angles (Prop. XVII. B. III.), and the exterior angle AFD is equal to the interior opposite angle E. Now this angle AFD, or AFC is equivalent to an angle at the circumference, subtended by the sum of the arcs AC, BG (Prop. XVI. Cor. 1. B. III.), and the angle E, to which it has just been shown to be equal, is subtended by the sum of the arcs AB, BG, and the arcs subtending equal angles are themselves equal; hence taking away the common arc BG, we have the arc AC equal to the arc AB, and consequently the chord AC is equal to the chord AB, which is impossible, except when the triangle ABC is isosceles.

Scholium.

1. It hence appears that the converse of proposition XXIX. fails only in the particular case of the isosceles triangle.

2. The above demonstration may be regarded as the analysis of the following problem, viz. To determine the triangle, the rectangle of whose sides is equivalent to the square of any line drawn from the vertex to the base, together with the rectangle of the parts into which it divides the base; and the above result shows that the triangle sought must be isosceles. By reversing the steps of the reasoning, and proceeding synthetically, it will result, that if the triangle be isosceles, the above property must always obtain. For suppose the triangle ABC to be isosceles, and let it be circumscribed by the circle ABGEC, and let any line AFG, be drawn from the vertex of the triangle to the circumference, while ADE bisects the angle A: join GE. Then since the arcs AB, AC are equal, it follows that the sum of the arcs AB, BG is equal to the sum of the arcs AC, BG, but these last are intercepted by the sides of the angle AFC: this angle is therefore equal to an angle at the circumference, which is subtended by an arc equal to this sum, that is to say, the angle AFC is equal to the angle AEG, consequently the angles GFD, DEG are together equal to two right angles; therefore (Prop. XVIII. B. III.) a circumference may be circumscribed about the quadrilateral FE, and consequently (Prop. XXVI. Cor. 2.), $AG \cdot AF = AE \cdot AD$, or which is the same thing, $AF^2 + AF \cdot FG = AD^2 + AD \cdot DE$; whence (Prop. XXIV.) $AF^2 + BF \cdot FC = AD^2 + BD \cdot DC = AB^2 + AC^2$ (Prop. XXIX.). Hence *in an isosceles triangle the square of a side is equivalent to the square of any line drawn from the vertex to the base, together with the rectangle of the parts into which it divides the base;* and here again we are very readily conducted to the property of the right angled triangle, demonstrated at proposition X. book 1.

For if ABD be a triangle right angled at D, and if DC be taken equal to BD, and AC be drawn, then AD bisects the angle A, and the base BC, of the isosceles triangle ABC; and, consequently, $AB^2 = AD^2 + BD^2$.

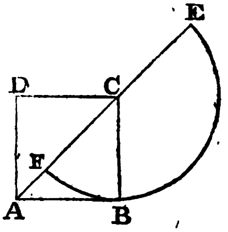
3. The student may exercise his ingenuity in demonstrating the above property of the isosceles triangle independently of any propositions besides those which the two first books furnish.

PROPOSITION XXXI. THEOREM.

The diagonal and side of a square are incommensurable.

Let ABCD be a square, the diagonal AC is incommensurable with its side AB.

From the point C as a centre, with the radius CB, describe the semicircle FBE.



Then, the angle B being right, AB is a tangent to the circumference; consequently (Prop. XXVI. Cor. 1.) AE : AB :: AB : AF; and, therefore (Prop. XIX. B. V.), AE, AB are incommensurable, and, consequently, AC, AB are also incommensurable; for if these had a common measure, the same also would measure their sum AE (Prop. XVII. B. V.), which has been proved to be incommensurable with AB; hence the diagonal of a square is incommensurable with its side.

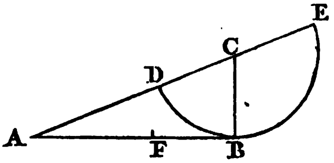
Scholium.

It appears from the above demonstration that it would be in vain to attempt to express accurately by numbers the side and diagonal of a square; a fact which might, indeed, have been inferred from the third corollary to Proposition X. Book I. For, representing the side of a square by unity, double the square of the side will be 2; and, consequently, the diagonal will be expressed by the square root of 2. Now $\sqrt{2}$ is a surd expression, that is to say, its numerical value can never be accurately found, although it may be approximated to sufficiently near for every practical purpose. This circumstance affords a striking instance of the insufficiency of numbers to answer rigorously all the purposes of geometry. We cannot, for instance, take upon ourselves to say that any two lines that may be promiscuously proposed, shall be susceptible of accurate numerical representation, without first inquiring whether these lines are commensurable or not; since, for aught we know to the contrary, one of the proposed lines may be equal to the side, and the other to the diagonal of the same square, or else they may be similarly related to each other. The reasonings of geometry, however, are quite independent of any proviso of this kind. That triangles and rhomboids of equal altitudes are to each other as their bases, is a truth which proposition I. of this book establishes as indisputably, when these bases are incommensurable, as when they are commensurable; if, indeed, it did not, that proposition, however completely it might satisfy the demands of practice, would, in a scientific point of view, be very defective; for, as the above proposition shows, it is possible for the bases of these triangles and rhomboids to be incommensurable. Hence the great impropriety of confounding in books on geometry the expressions *product* and *rectangle*, since the terms of a product *must* be commensurable, while the sides of a rectangle *may* be incommensurable. Whatever is shown to be true of the rectangle of two lines must necessarily be true of the product of the numbers representing its sides, in the particular case when those sides are commensurable, or capable of such numerical representation. But it is evident that we cannot, conversely, from this particular case infer the general proposition in which it is included, without violating one of the most obvious rules of logic.

PROPOSITION XXXII. PROBLEM.

To divide a given straight line into two parts, such that the greater part may be a mean proportional between the whole line and the other part.

Let AB be the proposed line. Draw the perpendicular BC equal to half AB, and from C as a centre, with the radius CB, describe the semicircle DBE, make AF equal to AD, and AB will be divided in F; so that AB : AF : : AF : FB.



For, since AB is perpendicular to CB, it is a tangent, and, consequently (Prop. XXVI. Cor. 1.) AE : AB : : AB : AD; therefore (Prop. XIII. B. V.) AE − AB : AB : : AB − AD : AD. But, by construction, AB = DE and AD = AF; so that in this last proportion the first and third terms are respectively the same as AF, FB; therefore putting these in their place, the proportion is AF : AB : : FB : AD, or AF; therefore, by inversion, AB : AF : : AF : FB.

Cor. Since AB = DE, the proportion AE : AB : : AB : AD furnishes us with the method of performing the following similar problem, viz. *To increase a given line, so that it may be a mean proportional between the whole and the part added*, nothing more being necessary, after having performed the above construction, than to add DA to AB.

Scholium.

Lines, divided as the above problem directs, are said to be divided in *extreme and mean proportion;* and it is obvious, from proposition XIX. book V., that a line so divided is incommensurable with its parts. Hence it appears that incommensurable lines may be found at pleasure, or that if any line be proposed, one incommensurable thereto may always be discovered; a fact which seems to illustrate in some measure the propriety of the remarks subjoined to the preceding proposition.

BOOK VII.

DEFINITIONS.

POLYGONS carry particular names according to the number of their sides, those of three and of four sides—triangles and quadrilaterals—have been already considered.

1. *Polygons* of five sides are called *pentagons*, those of six sides *hexagons*, those of seven sides *heptagons*, those of eight *octagons*, and so on.

2. Polygons, which are at once equilateral and equiangular, are called *regular polygons*.

PROPOSITION I. THEOREM.

Two regular polygons of the same number of sides are similar.

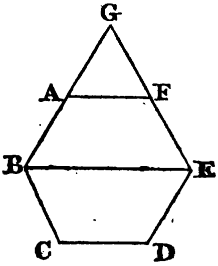
For the sides being equal in number, the angles are also equal in number; and the sum of the angles of the one polygon is equal to the sum of the angles of the other (Prop. XVII. B. I.); and, since the polygons are each equiangular, it follows that any angle in the one polygon is the same submultiple of their sum; and equi-submultiples of equal magnitudes being equal, the angles of the two polygons are all equal to each other; and it is obvious that the sides containing any angle in the one polygon are to each other as the sides containing any angle in the other, for each polygon is equilateral; therefore (Def. I. B. VI.) the polygons are similar.

PROPOSITION II. THEOREM.

A circle may be inscribed in, or circumscribed about, any regular polygon, and the circles so described have a common centre.

If the polygon ABCDEF be regular, then from a common centre a circle may be inscribed in, and circumscribed about, it.

This has already been shown to be true of the equilateral triangle and the square (Prop. XXIII. Schol., and Prop. XXV. Cor. B. IV.); we shall therefore consider the annexed polygon to have more than four sides.



Then, the angles of this polygon being each greater than a right angle (Prop. XVII. Cor. 4. B. I.), if the sides BA, EF be produced, they will meet as at G, forming the isosceles triangle GAF, and if BE be drawn, the triangle GBE will be also isosceles: hence, in the quadrilateral ABEF, the angles A, E are together equal to the angles F, B, and, consequently (Prop. XVIII. Cor. 1. B. III.), a circle may be circumscribed about it; therefore the points B, A, F, E all lie in the same circumference. By reasoning in a similar way it may be shown that the points A, F, E, D all lie in the same circumference, which circumference must be identical with the former (Prop. VIII. B. III.). Proceeding in this way it will appear that the same circumference must also pass through the next point C, and so on completely round the polygon.

Again, since the sides of the polygon are so many equal chords of the circumscribing circle, they must all be equally distant from the centre (Prop. VI. B. III.); and, consequently, the circumference described from the same centre, with this common distance as radius, must touch every side of the polygon.

Cor. 1. It has been shown that the triangles GAF, GBE, having the common angle G, are both isosceles; it, therefore, follows that in a regular polygon the diagonal, cutting off three sides, is parallel to the middle one, and, further, that the diagonal, cutting off five, or indeed any odd number of sides, must be parallel to the middle side; for if the sides intercepted by the diagonals, cutting off three and five sides, are parallel, the diagonals themselves must be parallel by Prop. XXIX. Book I., and if the intercepted sides are not parallel, they meet when produced, and form, with the diagonals, two similar isosceles triangles; so that in this case the diagonals are parallel, and in the same way is it to be shown that the diagonal, cutting off seven sides, is parallel to the middle one, and so on; therefore, generally, *the diagonal, which cuts off an uneven number of sides from a regular polygon, is parallel to the middle one of those sides.*

Cor. 2. *The angles formed at the centre of the circle, by lines drawn from it to the extremities of the sides of the polygon, are all equal,* being subtended by equal chords.

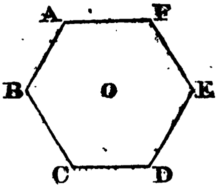
Scholium.

It may be remarked that in regular polygons the centre of the inscribed and circumscribed circles is called also the centre of the polygon, and that the perpendicular from the centre to a side, that is, the radius of the inscribed circle, is called the *apothem* of the polygon.

PROPOSITION III. THEOREM. (*Converse of Prop. II.*)

If from a common centre circles can be inscribed within, and circumscribed about a polygon, that polygon is regular.

Suppose that from the point O, as a centre, circles can be described in, and about, the polygon in the margin; this polygon is regular.

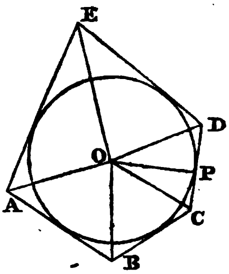


For, supposing these circles to be described, the inner one will touch all the sides of the polygon; these sides are, therefore, equally distant from its centre (Prop. IX. B. III.), and, consequently, being chords of the outer circle they are equal, and, therefore, include equal angles (Prop. XIV. Cor. 4. B. III.). Hence the polygon is at once equilateral and equiangular, that is (Def. 2.), it is regular.

PROPOSITION IV. THEOREM.

The surface of every polygon in which a circle may be inscribed, is equivalent to the rectangle of half the radius of that circle, and the perimeter of the polygon.

Let O be the centre of the circle inscribed in the polygon ABCD, &c. Draw from O lines to the extremities of the sides, thus dividing the polygon into as many triangles as it has sides.



Then, the common altitude of these triangles is the radius OP of the circle. Hence the surface of any one of them, OCD, for instance, is equivalent to the rectangle of half OP and CD (Prop. III. Cor. V. B. II.), and so of any other; therefore the sum of all the triangles, that is, the surface of the polygon, is equivalent to the rectangle of half the radius, and the whole perimeter of the polygon.

Cor. It has been shown (Prop. XXIII. B. IV.) that a circle may be inscribed in a triangle; consequently, *a triangle is equivalent to the rectangle of half the radius of the inscribed circle and its perimeter.*

Scholium.

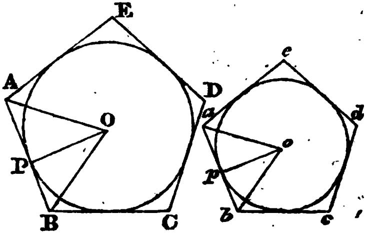
The above proposition is evidently true for all regular polygons (Prop. II.).

The converse of this proposition is as follows:—If the surface of a polygon be equivalent to the rectangle of its perimeter and another line, this line will be half the radius of the inscribed circle, which it is obvious has not place; for the surface of any polygon, whether it can circumscribe a circle or not, may always be represented by an equivalent rectangle, of which one side may be of any given length (Prop. XVII. Cor. B. IV.); so that when the perimeter is one side, the other side would be half the radius of the inscribed circle, whether the polygon admit of such inscription or not. But if the converse be enunciated thus:—If the surface of a polygon in which a circle may be inscribed be equivalent to a rectangle, of which one side is the perimeter, then the other side will be half the radius of the inscribed circle, or if one side be half the radius of the inscribed circle, the other side will be the perimeter—its truth immediately follows from the proposition itself.

PROPOSITION V. THEOREM.

If within two similar polygons circles can be inscribed, the perimeters of the polygons are as the radii of the circles; or if circles can be circumscribed about two similar polygons, then, also, their perimeters are as the radii of these circles, and in each case the surfaces of the polygons are as the squares of the radii.

First, let the similar polygons in the margin admit of inscribed circles, whose centres are O, *o*—their perimeters will be as the radii OP, *op*. Draw OA, OB, and *oa*, *ob*, and let AB, *ab*; BC, *bc*, &c. be homologous sides.

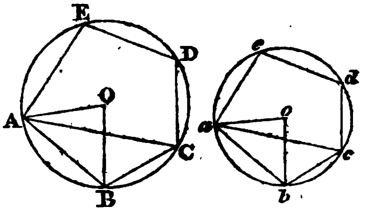


Now it has already been proved (Prop. XX. B. VI.) that the perimeters of similar polygons are as their homologous sides. It will, therefore, be necessary only to show that any two homologous sides, AB, *ab* for instance, are as the radii of the inscribed circles.

The angles A, B are bisected by the lines AO, BO; and the angles *a*, *b* are in like manner bisected by the lines *ao*, *bo* (Prop. XXVI. Cor. 5. B. VI.). Now the angles A, *a*; B, *b* are supposed to be respectively equal; and, therefore, the triangles OAB, *oab* are similar, and, consequently (Prop. XIII. B. VI.), the base AB is to the base *ab* as the perpendicular OP is to the perpendicular *op*, a proportion which, as before remarked, sufficiently establishes the theorem. Hence the perimeters are to each other as the radii of the inscribed circles. But (Prop. XXI. B. VI.) the surfaces of the polygons are to each other as the squares of the homologous sides AB, *ab*; or BC, *bc*, and these are as the squares of the radii (Prop. IV. Cor. 1. B. VI.), consequently the surfaces are to each other as the squares of the radii.

Next, let the similar polygons be circumscribed by circles, the centres being O, *o*, and let OA, OB, CA, CB be drawn in the one, and *oa*, *ob*, *ca*, *cb* in the other.

Then, since the polygons are similar, the triangles CAB, *cab* are similar (Prop. XIX. B. VI.); therefore the angles



ACB, *acb* are equal; but the angle O is double the angle ACB, and the angle *o* double the angle *acb*; therefore the angles O, *o* are equal, and, consequently, the isosceles triangles OAB, *oab* are similar; and therefore,

$$AB : ab :: OA : oa.$$

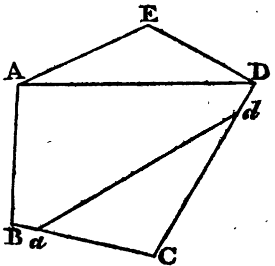
Hence the perimeters are in this case as the radii of the circumscribed circles; and it moreover follows, as above, that the surfaces are as the squares of the radii.

Cor. About, or within a circle, a polygon may be described similar to any inscribed or circumscribed polygon, by making angles at the centre of the proposed circle respectively equal to those which the sides of the polygon subtend, at the centre of its inscribed or circumscribed circle.

Scholium.

The above proposition is evidently true of all regular polygons of the same number of sides.

The converse of this proposition is not true. It may be enunciated thus:—If the perimeters of two polygons are to each other as the lines R, R', and their surfaces as R^2, R'^2, then R, R' must be radii of either the inscribed or circumscribed circles. That this is not true will be immediately perceived from considering that, although the polygons may be inscriptible or circumscriptible, R, R' may be any two lines, which are to each other as the radii of the inscribed or circumscribed circles. But it will not follow even that these polygons are similar, for two dissimilar polygons may exist, which shall, nevertheless, have their perimeters and surfaces respectively equivalent. To give a very simple illustration of this, let the polygon ABCDE be proposed, and let the distance of two points in its perimeter as A, D be equal to the distance of two other points as *a*, *d*. Then it is obvious that if portions of the polygon be cut off by the lines AD, *ad*, and then each portion added to the part whence the other was taken; so that AD may occupy the place of *ad*,

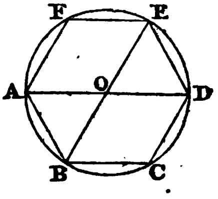


while *ad* occupies the place of AD; it will be evident that, although the perimeter and surface of the polygon remain unaltered by this transposition of parts, the form of the polygon will be changed, that is, it will be dissimilar to the original. Hence it appears that whatever relation may exist between the perimeters of two polygons, or between their surfaces, we shall be unable to infer, from such relation, either their similarity or their capability of being inscribed in, or circumscribed about, a circle. We have already seen that two polygons may be equivalent in surface and perimeter, without warranting such inference; and further yet, two polygons may exist which shall have the same number of sides, be equivalent both in perimeter and surface, both admit of inscription in the same circle, and yet be dissimilar. For an inscribed polygon is composed of certain isosceles triangles, whose vertices are all situated at the centre of the circle, and whatever be the order of the sides of the polygon round the circumference, this polygon will still be composed of the same triangles, and its perimeter will remain unaltered. Now supposing the sides to be all unequal, it is plain that every transposition of them will present a dissimilar polygon:—The triangle is the only exception to this. We can, however, assert thus much, that two inscribed polygons, whose sides are equal in length and number, must be equivalent in surface, and must also be contained in equal circles. For the one polygon may have its sides ranged round the circumference in the same order as the respectively equal sides of the other polygon, and thus become identical with it, preserving, as shown above, the same surface, and the same perimeter. But only one circle can be described about the same polygon, or which amounts to the same thing, only equal circles can circumscribe equal polygons; hence the truth of the above assertion is established. Again, all polygons circumscribed about equal circles and having equal perimeters, however they may differ as to the number of their sides, must be equivalent in surface; for the surface of each is equivalent to the rectangle contained by half the radius of the circle and its perimeter; and, by hypothesis, the circles and perimeters are respectively equal. We may, moreover, infer that polygons, equivalent in surface, circumscribed about equal circles, are also equal in perimeter.

PROPOSITION VI. THEOREM.

The side of a regular hexagon inscribed in a circle, is equal to the radius of that circle.

Let ABCDEF be a regular hexagon inscribed in a circle, the centre of which is O, then a side as BC will be equal to the radius OD.



For draw AD:—Then, since the arcs subtended by equal chords are equal, it follows that the three arcs AF, FE, ED are together equal to the three arcs AB, BC, CD; that is, the diagonal AD is a diameter of the circle. For similar reasons the diagonal BE is also a diameter. Now, AD is parallel to BC, and BE to CD (Prop. II. Cor. 1.); so that BD is a rhomboid, and, consequently, BC is equal to OD.

Scholium.

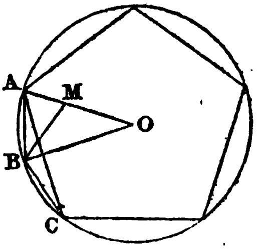
Hence, in order to inscribe a regular hexagon in a given circle, nothing more is required than to repeat the radius of the circle round the circumference. By joining the alternate points A, C; C, E; E, A, an equilateral triangle will be inscribed in the circle.

PROPOSITION VII. PROBLEM.

In a given circle to inscribe a regular decagon, and then a regular pentagon.

Let O be the centre, and OA the radius of the given circle; it is required to inscribe in this circle a regular polygon of ten sides.

Divide OA in extreme and mean proportion (Prop. XXXII. B. VI.); let M be the point of division; then a chord AB equal to OM, the greater part, will be the side of a regular decagon; so that by applying it ten times round the circumference, the required polygon will be inscribed.



For, join MB:—Then, by construction, AO : OM :: OM : AM; but AB=OM; therefore AO : AB :: AB : AM : hence (Prop. XI. B. VI.) the triangles ABO, AMB are similar, and, therefore, since ABO is isosceles, AMB must be isosceles; consequently AB=MB, therefore OM=MB; so that the triangle BMO also is isosceles, and the exterior angle AMB is therefore double the interior angle O; and the angle MAB being equal to the angle AMB, it follows that, in the isosceles triangle OAB, each of the angles A, B at the base, is double the angle O at the vertex, and as all three amount to two right angles, the angle O must be the fifth part of two right angles, or the tenth part of four right angles; consequently ten of these angles may be ranged round the point O; and as they are subtended by equal chords and equal arcs, the chord of the arc AB may be applied exactly ten times round the circumference, forming the inscribed decagon required.

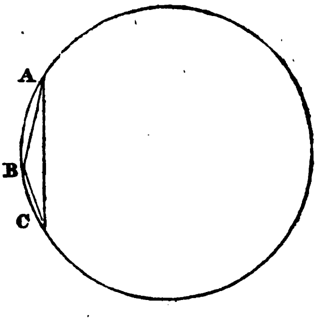
If the alternate corners of the decagon be joined, an inscribed regular pentagon will obviously be formed.

PROPOSITION VIII. PROBLEM.

In a given circle, to inscribe a regular pentedecagon, or polygon of fifteen sides.

Let AB be a side of the inscribed regular decagon, found by last proposition; and let AC be a side of a regular inscribed hexagon; that is, apply the radius from A to C (Prop. VI.). Join BC, then BC will be a side of the polygon required.

For AB cuts off an arc equal to a tenth part of the circumference; and AC subtends an arc equal to a sixth of the circumference; the difference of these arcs, therefore, is a fifteenth part of the circumference; and as equal arcs are subtended by equal chords, it follows that the chord BC may be applied exactly fifteen times round the circumference, thus forming a regular pentedecagon.



Scholium.

Since the perpendicular from the centre to a chord bisects the subtended arc, it is very easy, from having an inscribed polygon given, to insert another of double the number of sides. Thus, from having an inscribed square (Prop. XXIV. B. 4), we may inscribe in succession polygons of 8, 16, 32, 64, &c. sides; from the hexagon may be formed polygons of 12, 24, 48, 96, &c. sides; from the decagon polygons of 20, 40, 80,

&c. sides, and from the pentedecagon we may inscribe polygons of 30, 60, 120, &c. sides; and it is plain that each polygon will exceed the preceding in surface.

It is obvious that any regular polygon whatever might be inscribed in a circle, provided that its circumference could be divided into any proposed number of equal parts; but such division of the circumference, like the trisection of an angle, which, indeed, depends on it, is a problem which has not yet been effected. There are no means of inscribing in a circle a regular heptagon, or which is the same thing, the circumference of a circle cannot be divided into seven equal parts, by any method hitherto discovered. Indeed the polygons above noticed were, till about a quarter of a century ago, supposed to include all that could admit of inscription in a circle; but in 1801 a work was published by *M. Gauss* of Göttingen, (and afterward translated into French by *M. Delisle*, under the title of *Recherches Arithmetiques*), containing the curious discovery that the circumference of a circle could be divided into any number of equal parts capable of being expressed by the formula 2^n+1, provided it be a prime number, that is, a number that cannot be resolved into factors. The number 3 is the simplest of this kind, it being the value of the above formula when $n=1$; the next prime number is 5, and this also is contained in the formula. But polygons of 3 and of 5 sides have already been inscribed. The next prime number expressed by the formula is 17, so that it is possible to inscribe a seventeen sided polygon in a circle. The investigation of Gauss's theorem, although it establishes the above geometrical fact, depends upon the theory of algebraical equations, and involves other considerations of a nature that do not enter into elements of geometry; we must, therefore, content ourselves with merely alluding to it *.

PROPOSITION IX. PROBLEM.

An inscribed regular polygon being given to circumscribe a similar polygon about the circle; and, *conversely*, from having a circumscribed regular polygon to form the similar inscribed one.

Let *abcd*, &c. be a regular inscribed polygon; it is required to describe a similar polygon about the circle.

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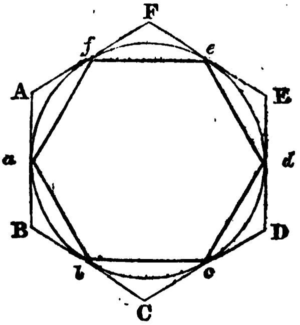
* On this subject the student may consult Barlow's Theory of Numbers.



At each of the points *a*, *b*, *c*, *d*, &c. draw tangents to the circle, and they will form the polygon ABCD, &c. similar to the polygon *abcd*, &c.

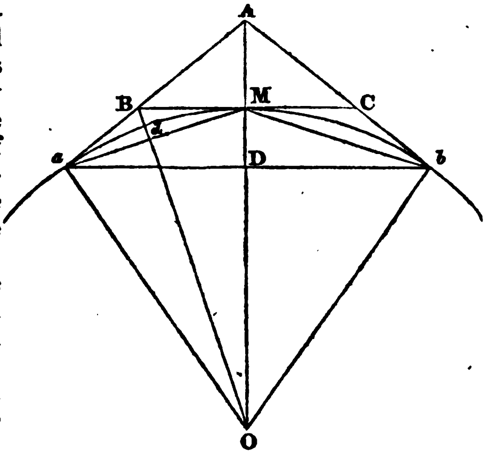
For, in the first place, there are as many tangents as the inscribed polygon has sides, and those drawn through the extremities of the same chord meet, otherwise the chord would be a diameter (Prop. IX. Cor. 3. B. III.). Next, the angles formed by these tangents and chords are all equal to each other, for their sides include equal arcs (Prop. XV. B. III.). Hence the triangles *f*A*a*, *a*B*b*, *b*C*c*, &c. are isosceles, and they have equal bases *fa*, *ab*, *bc*, &c.; therefore these triangles are equal, and, consequently, the angles A, B, C, D, &c. are equal, and so are their including sides: therefore the polygon A B C D &c. is regular; and it has the same number of sides as the polygon *a b c d* &c., it is, therefore, similar to it (Prop. I.).

Conversely. Let the circumscribed polygon A B C D &c. be given, then if the successive points of contact *a*, *b*, *c*, *d*, &c. be joined, a similar polygon will be inscribed in the circle:—For the angles A, B, C, &c. are equal, as also the sides *a*B, B*b*, *b*C, C*c*, &c., each being half a side of the polygon; consequently the sides *ab*, *bc*, *cd*, &c. are equal, and equal chords include equal angles (Prop. XV. Cor. IV. B. 3.); they, therefore, form a regular polygon, and as the sides are the same in number as those of the circumscribed polygon, it is similar to it.

Scholium.

1. It was remarked in the scholium to the preceding proposition that, from having an inscribed regular polygon, we might easily form another of double the number of sides. It may be, in like manner, here observed that, from having a circumscribed regular polygon, we may readily derive another of double the number of sides, nothing more being necessary than to draw tangents to the points of bisection of the arcs intercepted by the sides of the proposed polygon, limiting these tangents by those sides; and it is plain that each of the polygons so formed will be less in surface than the preceding, being entirely comprehended within it.

Let *ab* be a side of an inscribed polygon, and if *a*A, *b*A be tangents to the circle at the points *a*, *b*, each will be one half



of the side of the similar circumscribed polygon, or which is the same thing, they will together be equal to the side of a circumscribed polygon, similar to the inscribed one whose side is *ab*. Let M be the middle of the intercepted arc, and draw M*a*, M*b*, and the tangent BMC, then *a*M, M*b* will be two consecutive sides of an inscribed polygon, having double the number of sides that the polygon has whose side is *ab;* and, consequently, BC being a tangent at M, meeting the tangents at *a* and *b* must, by the proposition, be the side of a polygon, having double the number of sides that the polygon has, whose side is *ab.*

2. If polygons be thus successively circumscribed about the circle, their perimeters will decrease as the number of sides increase. For BC is less than AB+AC, and, consequently, $a\text{B}+\text{BC}+\text{C}b<a\text{A}+\text{A}b$; now, $a\text{B}+\text{C}b=\text{BC}$; and $a\text{A}+\text{A}b$ is equal to a side of the first circumscribed polygon: hence two sides of the second circumscribed polygon are together less than one side of the first; and, therefore, the whole perimeter of the second is less than that of the first. It is obvious that with respect to the inscribed polygons, the perimeters increase in the same circumstance, thus:—The two sides *a*M, M*b* being together longer than *ab*, it follows that the perimeter of the second inscribed polygon exceeds that of the first.

The successive circumscribed polygons that we have been considering continually approach nearer and nearer towards coincidence with the circle; for OB is nearer an equality to the radius O*a* of the circle than OA, because the distance *a*B is less than the distance *a*A is; and in every succeeding polygon the difference between the radius of the circle and the distance of the centre from the remotest points in the perimeter will, in like manner, perpetually diminish; so that the perimeters continually approach towards coincidence with the circumference, and we have already seen that these perimeters continually *diminish.*

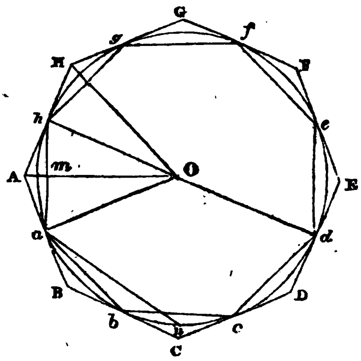
Now it is plain that if a series of magnitudes continually approach nearer and nearer towards coincidence with any proposed magnitude, and at the same time continually diminish; the magnitude to which they approach must be smaller than either of the approaching terms; we are, therefore, warranted in asserting that *the circumference of a circle is a shorter line than the perimeter of any circumscribed polygon.*

In a similar manner, by considering the successive inscribed polygons, it appears that *they* also continually approach towards coincidence with the circle; for O*d* is nearer an equality to the radius than OD, since the chord *a*M is shorter than *ab* (Prop. VII. B. III.); so that in each succeeding polygon the perimeter approaches nearer to coincidence with the circumference, and it has been shown that these perimeters successively *increase:* hence we may infer that *the circumference of a circle is a longer line than the perimeter of any inscribed polygon.*

PROPOSITION X. THEOREM.

Two polygons may be formed, the one within, and the other about a circle, that shall differ from each other by less than any assigned magnitude however small.

Let M represent any assigned surface, it is to be shown that two polygons may be described, the one within, and the other about the circle, whose centre is O, which will differ from each other by a magnitude less than M.



Let N be the side of a square, whose surface is less than the surface M, and inscribe in the circle a chord *an* equal to the line N. Then, by the methods already explained, inscribe in the circle a square, a hexagon, or indeed any regular polygon; let the arcs which its sides subtend be bisected, the chords of the half arcs will be the sides of a regular polygon, having double the number of sides: let, now, the arcs subtended by the sides of this second polygon be in like manner bisected, the chords will form a third polygon, having double the number of sides that the second has. Continue these successive bisections till the arcs become so small as to be each less than the arc an, their chords forming the inscribed polygon $abcd$, &c. Circumscribe the circle with a similar polygon ABCD, &c., then this last will exceed the former by a magnitude less than the proposed magnitude M. From the centre O draw the lines Oa, OA, Oh, OH, and produce hO to d; then the polygon $abcd$, &c. is composed of as many triangles equal to Oah as the polygon has sides, and in like manner the polygon ABCD, &c. is composed of as many triangles equal to OAH as this polygon has sides, and as the polygons have each the same number of sides, the inscribed is the same multiple of the triangle Oah that the circumscribed is of the triangle OAH. Now the triangle Omh is half the triangle Oah, in like manner the triangle OhA is half the triangle OAH; hence the inscribed polygon is the same multiple of Omh that the circumscribed polygon is of OhA, and consequently,

$$Omh : OhA :: ins.\ pol. : circ.\ pol.,$$

whence (Prop. XIII. B. V.)

$$OhA : OhA - Omh :: circ.\ pol. : circ.\ pol. - ins.\ pol.,$$

that is,

$$OhA : Amh :: circ.\ pol. :: circ.\ pol. - ins.\ pol.$$

Now OhA is a right angle, and since AO bisects the angle A of the isosceles triangle aAh, it is perpendicular to ah; therefore the triangles OhA, Amh are similar, consequently, (Prop. XXI. B. VI.).

$$OhA : Amh :: Oh^2 : hm^2 :: hd^2 : ha^2, \text{ whence}$$

$$hd^2 : ha^2 :: circ.\ pol. :: circ.\ pol. - ins.\ pol.$$

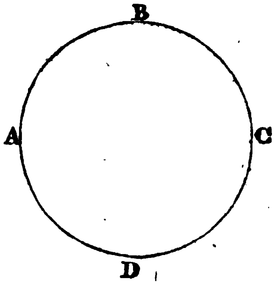
Now a circumscribed square, that is to say, hd^2 is greater than the polygon ABCD, &c., since the surfaces of circumscribed polygons diminish as their sides increase in number, so that in the last proportion the first antecedent is greater than the second, consequently, the first consequent is greater than the second, that is, the excess of the circumscribed polygon above the inscribed is less than ha^2, and therefore less than N^2 or than M,

Cor. As the circle is obviously greater than any inscribed polygon and less than any circumscribed one, it follows that *a polygon may be inscribed or circumscribed, which will differ from the circle by less than any assignable magnitude.*

PROPOSITION XI. THEOREM.

A circle is equivalent to the rectangle contained by lines equal to the radius and half the circumference.

Let us represent the rectangle of the radius and semi-circumference of the circle ABCD by P: we are to show that this rectangle is equal in surface to the circle.



If the rectangle P be not equivalent to the circle it must be either greater or less. Suppose it to be greater, and let us represent the excess by Q.

Then, by the corollary to last proposition, a polygon may be circumscribed about the circle, which shall differ therefrom by a magnitude less than Q, and must consequently be less than the rectangle P. But every circumscribed polygon is equivalent to the rectangle of the radius and half its perimeter, and the perimeter exceeds the circumference of the circle, consequently the rectangle of the radius of the circle and semi-perimeter of the polygon must be greater than P, the rectangle of the same radius and semi-circumference of the circle; but it was shown above to be less, which is absurd; hence the hypothesis that P is greater than the circle, is false.

But suppose the rectangle P is less than the circle, and let us represent the defect by the same letter Q.

Then, by the same corollary, a polygon may be inscribed in the circle, which shall differ from it by a magnitude less than Q, and must consequently be greater than the rectangle P. But every inscribed polygon is equivalent to the rectangle of its apothem and half its perimeter, and the apothem is less than the radius of the circle, and the perimeter less than the circumference; consequently the rectangle of the apothem and semi-perimeter of the polygon must be less than P, the rectangle of the radius and semi-circumference of the circle; but it was shown above to be greater, which is absurd; hence the second hypothesis also is false.

As therefore the circle can be neither greater nor less than the rectangle P, it must necessarily be equivalent to it.

PROPOSITION XII. THEOREM.

Circles are to each other as the squares of their radii.

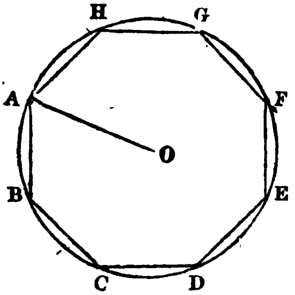
Let the circles ABCD, *abcd*, be compared, we shall have the proportion

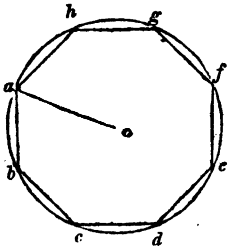
$$AO^2 : ao^2 :: circ.\ ABCD : circ.\ abcd.$$

For if this proportion has not place let there be

$$AO^2 : ao^2 :: \mathit{circ.}\ ABCD : P,$$

P being some magnitude either greater or less than the circle *abcd*. Suppose it to be less, and let us represent the defect by Q. Then (Prop. X. Cor.) a polygon may be inscribed in the circle *abcd*, which shall differ from it by a magnitude less than Q, and will therefore exceed the magnitude P. Let *abcde*, &c. be such a polygon, and describe a similar polygon ABCDE, &c. in the other circle. Then (Prop. V.)





$$AO^2 : ao^2 :: \mathit{pol.}\ ABCDE, \&c. : \mathit{pol.}\ abcde, \&c.$$

combining this with the proportion above we have

$$\mathit{circ.}\ ABCD : P :: \mathit{pol.}\ ABCDE, \&c. : \mathit{pol.}\ abcde, \&c.$$

Now in this proportion the first antecedent is greater than the second, consequently the first consequent is greater than the second, that is, P is greater than the polygon *abcde*, &c., but it has been shown to be less, which is absurd. Therefore P cannot be less than the circle *abcd*.

But suppose that P is greater than the circle *abcd*. Then, still representing the difference by Q, a polygon may be circumscribed about the circle *abcd*, which shall differ from it by a magnitude less than Q, or be less than P. Suppose such a polygon to be described, and that a similar one is formed about the circle ABCD, then these polygons being to each other as the squares of the radii of their respective circles, it will evidently result by combining, as in the preceding case, this proposition with that advanced in the hypothesis, that the circle ABCD is to P, as the polygon about this circle to the polygon about the other; in which proportion the first antecedent is less than the second, and consequently the first consequent is less than the second, that is, P is less than the polygon circumscribed about the circle *abcd*; but it was shown above to be greater, which is impossible. Hence, P can neither be less nor greater than the circle *abcd*, consequently it must be equal to it, and therefore,

$$AO^2 : ao^2 :: circ.\ ABCD : circ.\ abcd.$$

Cor. 1. Since every circle is equivalent to the rectangle of its radius and half its circumference, the above proportion may be expressed thus:

$$AO^2 : ao^2 :: AO \cdot \tfrac{1}{2} ABCD : ao \cdot \tfrac{1}{2} abcd,$$

whence (Prop. I. Cor. B. VI.)

$$AO : ao :: \tfrac{1}{2}\ ABCD : \tfrac{1}{2}\ abcd.$$

Consequently *the circumferences of circles are to each other as their radii*, and therefore *their surfaces are as the squares of the circumferences.*

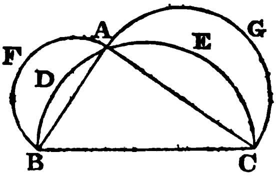
Cor. 2. It follows also, that *similar arcs are to each other as the radii of the circles to which they belong*, for they subtend equal angles at the centres (Def. 2. B. VI.) and each angle is to four right angles as the arc which subtends it is to the whole circumference (Prop. XXIII. Cor. 1. B. VI.); consequently the one arc is to the whole circumference, of which it forms part, as the other arc to the circumference of which it is part: and as the circumferences are as the radii, we have alternately the one arc to the other as the radius of the former to that of the latter.

Cor. 3. Therefore also *similar sectors are to each other as the squares of their radii*, for each sector is to the circle as the arc to the circumference (Prop. XXIII. Cor. 2. B. VI); consequently the one sector is to its circle as the other sector to its circle: and as the circles are as the squares of the radii, we have alternately the one sector to the other as the square of the radius of the former to the square of that of the latter.

Cor. 4. It readily follows that *similar segments are also as the squares of the radii*, for they result from similar sectors, by taking away from each the triangle formed by the chord and radii, which triangles being similar, are also to each other as the squares of the radii; therefore the sectors and triangles being proportional, it follows (Prop. XI. B. V.) that the segments also are as the sectors or as the squares of the radii, or indeed as the squares of their chords.

Scholium.

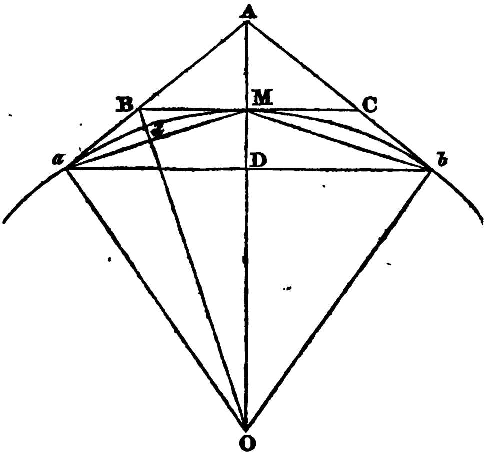
From this proposition and its corollaries may easily be derived an extension of the property of the right angled triangle which forms proposition XXII. of Book VI., for it may now be proved that if circles be described about the three sides taken as diameters, or if similar sectors or segments be formed on the sides, it will always result that the figure on the hypothenuse will be equivalent to both those on the sides. By turning to the proposition alluded to we shall find that the reasoning there employed applies equally to the demonstration of this property, and therefore it need not be here repeated. Another very remarkable property arises out of that just mentioned, and which must not remain unnoticed. Let semicircles be described on the hypothenuse BC, and on the sides AB, AC of the right angled triangle ABC; then since the semicircle BDAC is equivalent to both the semicircles BFA, AGC, it follows that if the common segments BDA, AEC be taken away, there will remain the triangle ABC equivalent to the two circular spaces or *lunes* BFAD, AGCE.



PROPOSITION XIII. PROBLEM.

The surface of a regular inscribed polygon and that of a similar circumscribed polygon being given, to find the surfaces of regular inscribed and circumscribed polygons of double the number of sides.

Let *ab* be a side of the given inscribed polygon, then the tangents *a*A, *b*A will each be half the side of the similar circumscribed polygon: the chords *a*M, *b*M to the middle of the arc *a*M*b* will be sides of an inscribed polygon of double the number of sides; and lastly, the tangent BMC will be the side of a circumscribed polygon similar to this last. All this is evident from proposition IX.



Let us now, in order to avoid confusion, denote the inscribed polygon whose side is ab by p, the corresponding circumscribed polygon by P; the inscribed polygon of double the number of sides by p', and the similar circumscribed polygon by P′. Then it is plain that the space Oad is the same part of p that OaA is of P, that OaM is of p', and that OaBM is of P′; for each of these spaces requires to be repeated the same number of times to complete the several polygons to which they respectively belong. Hence then, and because magnitudes are as their like multiples, it follows that whatever relations are shown to exist among these spaces will be true also of the respective polygons of which they form part. Now the right angled triangles ODa, OAa, BMA are similar: the two first furnish the proportion OD : Oa :: Oa : OA, or which is the same thing, OD : OM :: OM : OA; and consequently, since triangles of the same altitude are as their bases, it follows that

$$\mathrm{OD}a : \mathrm{OM}a :: \mathrm{OM}a : \mathrm{OA}a,$$

that is, the triangle OMa is a mean between ODa and OAa; consequently the polygon p' is a mean between the polygons p and P.

Again, the similar triangles ODa, BMA give the proportion OD : Oa :: BM : BA, or which is the same thing, OD : OM :: aB : BA; and consequently, since triangles of the same altitude are as their bases, it follows that

$$\mathrm{OD}a : \mathrm{OM}a :: \mathrm{O}a\mathrm{B} : \mathrm{OBA}, \text{ therefore}$$

$$\mathrm{OD}a + \mathrm{OM}a : 2\mathrm{OD}a :: \mathrm{O}a\mathrm{B} + \mathrm{OBA} : 2\mathrm{O}a\mathrm{B},$$

consequently

$$p + p' : 2p :: \mathrm{P} : \mathrm{P}'.$$

Scholium.

It was proved in proposition XI. that a circle is equivalent to the rectangle contained by its radius, and a straight line equivalent to half its circumference. In order, therefore, to construct a rectangle equivalent to any given circle, it would only be necessary, from having the radius, to draw a straight line equal to half the circumference. But this is a problem that has never yet been effected, so that the equivalent rectangle remains still undetermined, and therefore the *quadrature of the circle*, as this problem is called, is not capable of being rigorously ascertained. This, however, is a circumstance little to be regretted, for it has been shown (Prop. X. Cor.) that polygons may be inscribed in, and circumscribed about, a circle that shall approach so near to coincidence with it as to differ from it by a magnitude less than any that can be possibly assigned: a degree of approximation obviously equivalent to perfect accuracy, since no magnitude can be found sufficiently small to denote its difference therefrom. The principal object of inquiry then should be, at least in a practical point of view, how we may most expeditiously carry on the approximation alluded to; and the problem above furnishes us with one of the best elementary methods for this purpose that can be given.

Let us represent the radius of the circle by 1, and let the first inscribed and circumscribed polygons be squares: the side of the former will be $\sqrt{2}$, and that of the latter 2, so that the surface of the former will be 2, and that of the latter 4. Now it has been proved in the proposition that the surface of the inscribed octagon, or, as we have denoted it, p', will be a mean between the two squares p and P, so that $p' = \sqrt{8} = 2{\cdot}8284271$. Also from the proportion $p + p' : 2p :: \mathrm{P} : \mathrm{P}'$ we obtain the numerical value of the circumscribed octagon, that is, $\mathrm{P}' = \frac{2p \cdot \mathrm{P}}{p + p'} = \frac{16}{2 + \sqrt{8}} = 3{\cdot}3137085$. Having thus obtained numerical expressions for the inscribed and circumscribed polygons of eight sides, we may from these, by an application of the same two proportions in a similar way, determine the surfaces of those of sixteen sides, and thence the surfaces of polygons of thirty-two sides, and so on till we arrive at an inscribed and circumscribed polygon, differing from each other, and consequently from the circle, so little that either may be considered as equivalent to it. The subjoined table exhibits the *area*, or numerical expression for the surface, of each succeeding polygon carried to seven places of decimals.

Number of sides.	Area of the inscribed polygon.	Area of the circumscribed polygon.
4	2·0000000	4·0000000
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
128	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926

It appears then that the inscribed and circumscribed polygons of 32768 sides differ so little from each other that the numerical value of each, as far as seven places of decimals, is absolutely the same: and as the circle is between the two, it cannot, strictly speaking, differ from either so much as they do from each other; so that the number 3·1415926 expresses the area of a circle whose radius is 1, correctly, as far as seven places of decimals. We may, therefore, conclude that were the absolute quadrature of the circle attainable, it would exactly coincide with the above number, as far at least as the seventh decimal place, which is an extent even beyond what the most delicate numerical calculations are ever likely to require. Were it necessary, however, the approximation might be continued to double the number of decimals: it has indeed been carried by some to a much greater length than this. *Ludolph van Ceulen* had the patience to extend the approximation as far as the thirty-sixth place of decimals, by a method somewhat different indeed from that above described, but requiring an equal degree of labour and attention. Since his time the quadrature of the circle has been approached still nearer by other methods. An infinite series was discovered by *Machin*, by which he reached the quadrature as far as the 100th place of decimals, and which proved to be

3·1415926535,8979323846,2643383279,5028841971,
6939937510,5820974944,5923078164,0628620899,
8628034825,3421170679;

and even this number has been extended by later mathematicians thirty or forty figures further.

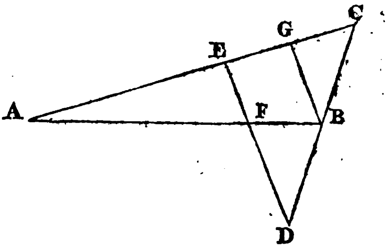
Having then found the numerical expression for the surface of a circle whose radius is 1, we readily find the area of any circle whatever; for since the surfaces are as the squares of the radii, we have only to multiply the square of the radius of any proposed circle by the number 3·14159, &c., and the product will be the area. Also, since the surface of a circle is equivalent to half the circumference multiplied by the radius (Prop. XI.), it follows that when the radius is 1 the half circumference must be 3·14159, &c.; or since the circumferences of circles are as their radii, when the diameter is 1, the circumference will be 3·14159, &c., so that the circumference of any circle is found by multiplying its diameter by 3·14159, &c., or, as is usual, simply by 3·1416. For the ordinary purposes of mensuration the circumference will be determined with sufficient precision by multiplying the diameter by 22, and dividing the product by 7, which is the approximation discovered by *Archimedes*. The fraction $\frac{7}{22}$ is equal to 3·1428, and consequently the circumference, as determined by this last method, differs from the truth by rather more than a thousandth part of the diameter, which in most practical cases is too inconsiderable to deserve notice.

BOOK VIII.

PROPOSITION I. PROBLEM.

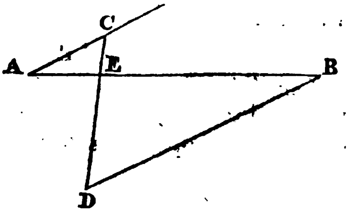
To divide a given straight line into any proposed number of equal parts.

Let it be proposed to divide the straight line AB into a certain number of equal parts.



From one extremity A draw an indefinite straight line AC, making any angle with AB, and upon it repeat one more than the proposed number of equal distances; then, supposing the last to terminate in C, and the last but one in G, the line AG will be divided into the same number of equal parts that AB is to be divided into; let the point E be at the distance of two of those parts from C, then if CBD be drawn, making BD=CB, and the points D,E be joined, the line AB will be divided in F; so that BF will be one of the required parts of AB.

For draw GB; then since CG=GE and CB=BD, the sides CE, CD of the triangle CED are divided proportionally by the line GB; therefore (Prop. V. B. VI.) GB is parallel to ED or EF: therefore, in the triangle AGB, we have the proportion AG : EG : : AB : FB, but AG is a given multiple of EG; therefore (Prop. XI. B. V.) AB is the same multiple of FB.



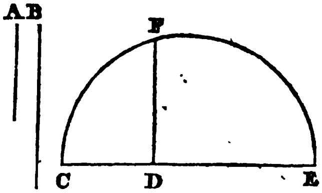
Otherwise as follows:—From one extremity A draw the indefinite straight line AC, making any angle with AB, and from the other extremity draw BD, making an equal angle with BA. Upon BD repeat the distance AC as many times, wanting one, as there are to be divisions of AB: draw CD, which will cut off from AB one of the required parts AE.

For, since the angles A, B are equal, the triangles EAC, EBD are similar; therefore AC : BD : : AE : EB; hence, whatever multiple BD is of AC, the same multiple is EB of AE, that is, AE is one of the proposed parts of AB.

PROPOSITION II. PROBLEM.

To find a mean proportional between two given straight lines.

Let the given lines be A and B.

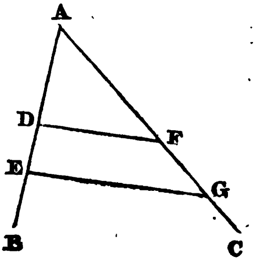


Draw a straight line CDE, making CD=A and DE=B, and upon CE describe a semicircle; then the perpendicular DF, drawn from D to the arc, will be a mean proportional between A and B. This is evident from Cor. 2. Prop. XXIV. Book VI.

PROPOSITION III. PROBLEM.

To find a fourth proportional to three given straight lines.

From any point A draw two straight lines AB, AC, forming any angle; and make AD, AE, AF respectively equal to the proposed lines; then it is required to find a fourth proportional to AD, AE, AF.



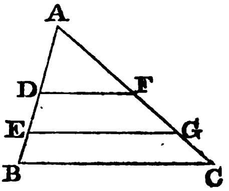
Join D, F, and parallel to DF draw EG; AG will be the fourth proportional required; for (Prop. V. Cor. B. VI.).

$$AD : AE :: AF : AG.$$

Cor. The same construction serves for finding a third proportional to two given lines, as A and B; this being the same as a fourth proportional to the three lines A, B, B.

PROPOSITION IV. PROBLEM.

To divide a given straight line AB into parts proportional to given lines.



Draw AC equal to the longest of the proposed lines, make AG equal to the line next in length, AF equal to the next, and so on:—join BC, and draw GE, FD, &c. parallel to BC, and the line AB will be divided by them as required; for these parallels cut the sides AB, AC of the triangle ABC proportionally (Prop. VI. Cor. B. VI.).

PROPOSITION V. PROBLEM.

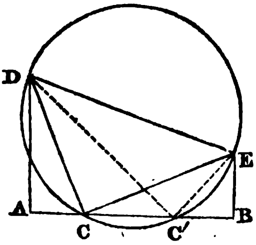
A straight line being given, to divide it so that the rectangle of the two parts may be equivalent to a given rectangle; or to prolong it, so that the rectangle contained by the whole line and the part added may be equivalent to a given rectangle.

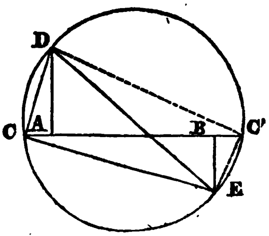
Let AB be the proposed straight line.

Then, from the extremities A, B draw the perpendiculars AD, BE, equal to the sides of the given rectangle, and both upon the same side of AB if it is to be divided, but one on each side if it is to be prolonged: draw DE, on which as a diameter describe a circle meeting AB, or its extension in the point C; AC and CB are the parts required.

For draw DC and CE.

Then, the angle DCE being contained in a semicircle, is a right angle (Prop. XIV. Cor. 3. B. III.); and, therefore, in





both cases of the problem, the angles ACD, BCE are together equal to a right angle. But the angles ACD, CDA are likewise together equal to a right angle; and, consequently, the angles BCE, CDA are equal. Wherefore the right angled triangles CBE, CAD are similar; whence AC : AD :: BE : CB, and, therefore,

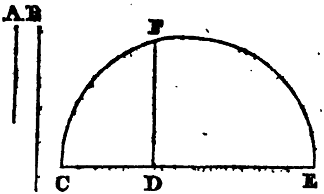
$$AC \cdot CB = AD \cdot BE.$$

Scholium.

It is obvious that in the second case of this problem, since a portion of the circle lies on each side of the line AB, the circumference must always intersect its extension in two points, C and C′. But, in the first case, the circle may either cut AB in two points C and C′, or touch it in a single point, which will hence mark a limitation of the problem. When the circle does not reach AB, the problem fails, thus intimating that the proposed line cannot be divided as required.

PROPOSITION VI. PROBLEM.

To construct a square that shall be equivalent to a given polygon.

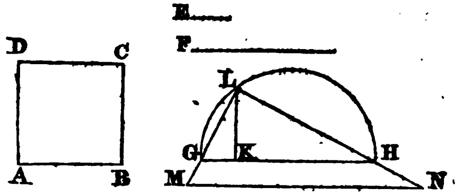


Reduce the proposed polygon to an equivalent rectangle (Prop. XVI. Cor. B. IV.), of which let A, B be the sides; draw a straight line CDE, making CD =A and DE=B: describe a semicircle on CE, and draw DF perpendicular to CE, terminating in the arc; DF will be the side of the square sought, for (Prop. XXIII. Cor. 2. B. VI.) $DF^2 = CD \cdot DE$.

PROPOSITION VII. PROBLEM.

To construct a square that shall be to a given square, AC, as the line E is to the line F.

Draw an indefinite straight line GH, upon which take GK =E and KH=F; describe on GH a semicircle, and draw the

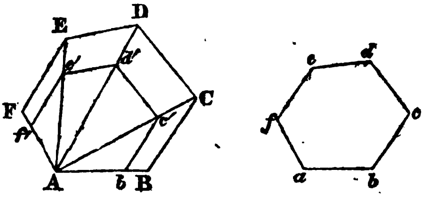


perpendicular KL. Through the points G, H draw the straight lines LM, LN, making the former equal to AB the side of the given square, and through the point M draw MN parallel to GH, then will LN be the side of the square sought.

For, since MN is parallel to GH, LM : LN :: LG : LH; consequently (Prop. IV. Cor. 1. B. VI.) $LM^2 : LN^2 :: LG^2 : LH^2$; but, since the triangle LGH is right angled, we have (Prop. XVII. Cor. B. VI.) $LG^2 : LH^2 :: GK : KH$; hence $LM^2 : LN^2 :: GK : KH$, but, by construction, GK=E and KH=F, also LM=AB; therefore the square described on AB is to that described on LN, as the line E is to the line F.

PROPOSITION VIII. PROBLEM.

Upon a given straight line ab to construct a polygon similar to a given polygon, ABCDEF.



In the given polygon draw the diagonals AC, AD, AE; and apply the given line ab to AB; making Ab' equal to it:—draw successively $b'c'$, $c'd'$, $d'e'$, $e'f'$ respectively parallel to BC, CD, DE, EF; then, from the points ab as centres, with radii equal to Ac', $b'c'$, describe arcs intersecting in c; also, from the centres a, c, with radii equal to Ad', $c'd'$, describe arcs intersecting in d; in like manner, from the centres a, d, with radii equal to Ae', $d'e'$, describe arcs intersecting in e; and, lastly, from the centres a, e, with radii equal to Af', $e'f'$, describe arcs intersecting in f; then, if the lines bc, cd, de, ef, fa be drawn, the polygon which they form will be similar to that proposed.

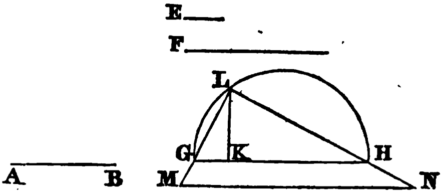
For the polygon $Ab'c'd'e'f'$ is similar to the polygon ABCDEF, since they are both composed of the same number of similar triangles, and the polygon $abcdef$ has been made equal to $a'b'c'd'e'f'$; hence the polygon on ab is similar to that on AB.

Scholium.

If ab is in the same straight line as AB, or if it is parallel to AB, the construction of this problem will be somewhat simplified. After having divided the proposed polygon into triangles as above, draw from the point *a* parallels to the diagonals AC, AD, AE; then from *b* draw *bc* parallel to BC, intersecting the first of these parallels in *c*; from *c* draw *cd* parallel to CD, intersecting the second parallel in *d*, and so on till the polygon on *ab* be completed.

PROPOSITION IX. PROBLEM.

A polygon being given, to construct a similar polygon that shall be to the former as the line E is to the line F.



Let AB be one side of the given polygon, and perform the same operation as in problem VII., that is to say, find a line, LN, such that $AB^2 : LN^2 :: E : F$, and LN will be the side of the required polygon which is homologous to the side AB; for similar polygons being to each other as the squares of their homologous sides, it follows that the polygon on AB is to the similar polygon on LN as E is to F; therefore it only remains to construct on LN, by the preceding proposition, a polygon similar to that on AB.

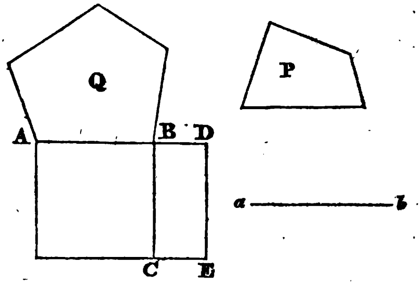
PROPOSITION X. PROBLEM.

Two similar polygons being given, to construct another similar polygon which shall be equivalent to either the sum or difference of the former.

The method of performing this problem immediately suggests itself from proposition XXII. B. VI. If a right angled triangle be constructed, having its perpendicular sides respectively equal to two homologous sides of the given polygon, the hypothenuse will be the homologous side of a third polygon similar to the former, and equal to their sum; and if two homologous sides of the given polygons be taken, the one for the hypothenuse and the other for a side of a right angled triangle, then the other side of this triangle will be the homologous side of a third polygon similar to the former, and equivalent to their difference. Hence, having found in this way a side of the required polygon, the construction is reduced to proposition VIII.

PROPOSITION XI. PROBLEM.

To construct a polygon, which shall be equivalent to a given polygon P, and similar to another polygon Q.

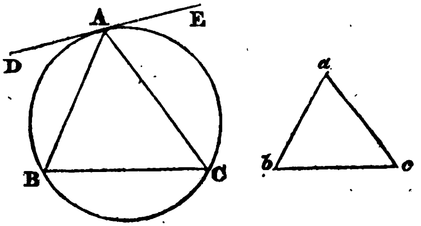


Upon AB, a side of the polygon Q, construct a rectangle AC equivalent to it (Prop. XVI. B. IV.), and on BC describe a rectangle BE equivalent to the other polygon P. Let *ab* be a mean proportional between AB, BD, then *ab* will be the side of a polygon similar to Q, and equivalent to P.

For, if a similar polygon *q* be constructed on *ab*, we shall have $AB^2 : ab^2 :: Q : q$, but, by construction, $ab^2 = AB \cdot BD$; therefore $AB^2 : AB \cdot BD :: Q : q$; consequently (Prop. I. Cor. B. VI.) $AB : BD :: Q : q$, and, therefore, also $AC : BE :: Q : q$; but $AC = Q$; therefore (Prop. IX. Cor. 2. B. V.) $BE = q$, that is $P = q$.

PROPOSITION XII. PROBLEM.

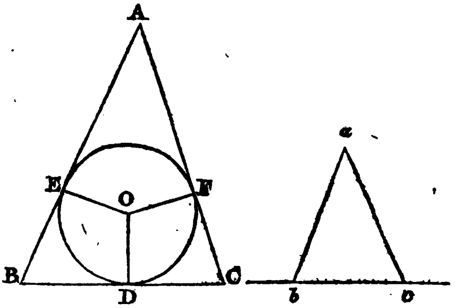
In a given circle to inscribe a triangle similar to a given triangle, *abc*.



Draw a tangent DAE to the circle at any point A in the circumference, and make the angle EAC equal to the angle *b*, and the angle DAB equal to the angle *c*. Draw BC, and the triangle ABC will be similar to the triangle *abc*. For (Prop. XV. B. III.) the angles B, C are respectively equal to the angles *b*, *c*; therefore the two triangles, being equiangular, are similar.

PROPOSITION XIII. PROBLEM.

About a given circle to circumscribe a triangle similar to a given triangle, *abc*.

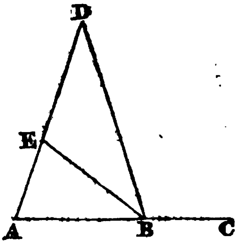


Produce a side *bc* of the triangle *abc*, and having drawn any radius OD, make the angles DOE, DOF equal respectively to the exterior angles *b*, *c*; then three tangents, drawn through the points D, E, F, will form a triangle ABC similar to the triangle *abc*.

For, in the quadrilateral ODBE, the angles O, B are together equivalent to two right angles (Prop. XVII. Cor. 3. B. I.): therefore, since the angle O has been made equal to the exterior angle *b*, it follows that the angle B is equal to the angle *abc*. In like manner it will appear that the angle C is equal to the angle *acb*: hence the triangles ABC, *abc* are similar.

PROPOSITION XIV. PROBLEM.

Upon a given base AB to construct an isosceles triangle, having each of the angles at the base double the vertical angle.



Produce AB till the rectangle AC·BC may be equal to the square of AB (Prop. XXXII. Cor. B. VI.), then, with the base AB and sides each equal to AC, construct the isosceles triangle DAB, and the angle A will be double the angle D.

For make DE = AB, or make AE = BC, and join EB.

Then, by construction, AD : AB :: AB : AE, for AE = BC; consequently the triangles DAB, BAE have a common angle A contained by proportional sides: hence (Prop. XI. B. VI.) they are similar, and, therefore, these triangles are both isosceles, for

DAB is isosceles by construction, so that AB=EB; but AB=DE; consequently DE=EB, and, therefore, the angle D is equal to the angle EBD: hence the exterior angle AEB is equal to double the angle D, but the angle A is equal to the angle AEB; therefore the angle A is double the angle D.

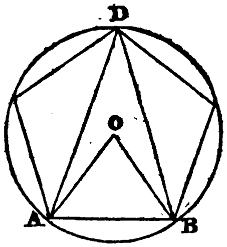
Scholium.

It is obvious that, in a triangle so constructed, the vertical angle is a fifth part of two right angles, and each angle at the base is two fifths of two right angles, or one fifth of four right angles.

PROPOSITION XV. PROBLEM.

Upon a given straight line to construct a regular pentagon.

Construct, first, upon AB an isosceles triangle DAB, having each of the angles at the base double the vertical angle; and about this triangle circumscribe a circle; then the line AB will be the side of the regular inscribed pentagon.

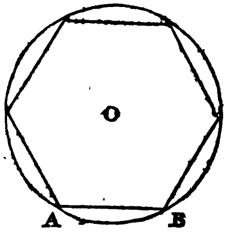


For, if the radii OA, OB be drawn, the angle O, being double the angle D, will be a fifth part of four right angles; consequently (Prop. XXIII. Cor. 1. B. VI.) the arc AB is the fifth part of the whole circumference, and, therefore, the chord AB is the side of a regular inscribed pentagon.

PROPOSITION XVI. PROBLEM.

Upon a given straight line AB to construct a regular hexagon.

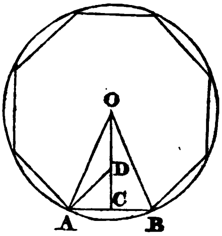
From the points A, B as centres, with radii equal to AB, describe arcs intersecting in O; and from O, with the same radius, describe a circle, then AB will be the side of the inscribed regular hexagon, as is manifest from proposition VI. book VII.



PROPOSITION XVII. PROBLEM.

Upon a given straight line AB to construct a regular octagon.

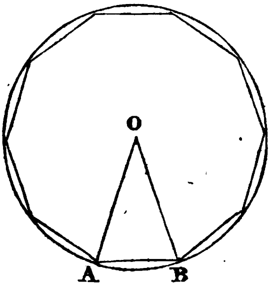
Bisect AB by the perpendicular CO, make CD=CA; draw DA, and make DO equal to it, then from the centre O, with the radius OA, describe a circle, and AB will be the side of the inscribed regular octagon.



Draw OB:—Then, in the right angled triangle ACD, because AC is equal to CD, the angle ADC must be one half a right angle, and it is equal to both the angles DAO, DOA, and these angles are themselves equal, because AD=DO; therefore the angle ADC is double the angle AOC, that is, it is equal to the angle AOB; this angle, therefore, is one half a right angle, or the eighth part of four right angles, so that the arc AB is the eighth part of the circumference; and, consequently, the chord AB is the side of the inscribed regular octagon.

PROPOSITION XVIII. PROBLEM.

Upon a given straight line AB to construct a regular decagon.



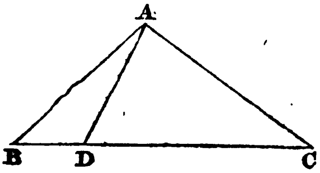
Construct upon the base AB an isosceles triangle, whose vertical angle O shall be half of the angle A, then from the centre O, with the radius OA, describe a circle, and AB will be the side of a regular decagon inscribed in that circle.

For the angle O is a tenth part of four right angles, and, therefore, the arc AB is a tenth part of the circumference; consequently the chord AB may be applied exactly ten times round the circumference, thus forming a regular decagon.

PROPOSITION XIX. PROBLEM.

To divide a triangle ABC into two parts by a line from A, the vertex of one of its angles, so that the parts may be to each other as a straight line M to another straight line N.

Divide BC into parts BD, DC proportional to M, N; draw the line AD, and the triangle ABC will be divided as required.



For since triangles of the same altitude are to each other as their bases, we have ABD : ADC :: BD : DC :: M : N.

Scholium.

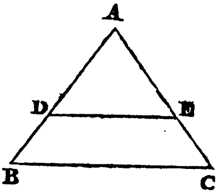
A triangle may evidently be divided into any number of parts proportional to given lines, by dividing the base in the same proportion.

PROPOSITION XX. PROBLEM.

To divide a triangle ABC into two parts by a line drawn parallel to a side BC, so that these parts may be to each other as two straight lines M, N.

As M+N is to N, so make AB^2 to AD^2 (Prop. VII.).

Draw DE parallel to BC, and the triangle is divided as required.

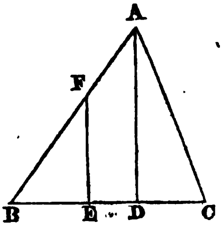


For the triangles ABC, ADE being similar, ABC : ADE :: AB^2 : AD^2; but M+N : N :: AB^2 : AD^2; therefore ABC : ADE :: M+N : N; consequently (Prop. XIII. B. V.) BDEC : ADE :: M : N.

PROPOSITION XXI. PROBLEM.

To divide a triangle ABC into two parts by a line perpendicular to the base, so that these parts may be to each other as two given lines M, N.

Draw the perpendicular AD, and as M+N is to N, so make



the square which is equivalent to BC·BD to BE²; then the perpendicular EF will divide the triangle as required.

For since the triangles ABC, FBE have the angle B in common, it follows (Prop. XVI. B. VI. Cor. 3.) that

$$ABC : FBE :: BC{\cdot}BD : BE^2;$$

but by construction,

$M+N : N :: BC{\cdot}BD : BE^2$; therefore

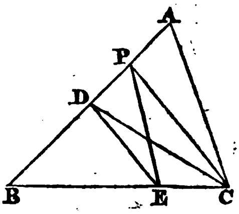
$$ABC : FBE :: M+N : N$$

consequently $AFEC : FBE :: M : N$.

PROPOSITION XXII. PROBLEM.

To divide a triangle into two parts by a line drawn from a given point P in one of its sides, so that the parts may be to each other as two given lines M, N.

Draw PC, and divide AB in D, so that AD is to DB as M is to N; draw DE parallel to PC, join PE, and the triangle will be divided by the line PE into the proposed parts.



For join DC; then because PC, DE are parallel, the triangles PDE, CDE are equal; to each add the triangle DEB, then PEB=DCB; and consequently, by taking each from the triangle ABC, there results the quadrilateral ACEP equivalent to the triangle ACD.

Now $ACD : DCB :: AD : DB :: M : N$;

consequently,

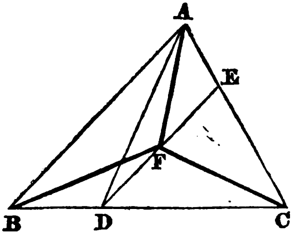
$$ACEP : PEB :: M : N.$$

Scholium.

The above operation suggests the method of dividing a triangle into any number of equal parts by lines drawn from a given point in one of its sides: for if AB be divided into equal parts, and lines be drawn from the points of division parallel to PC, they will intersect BC, and AC; and if from these several points of intersection lines be drawn to P, they will divide the triangle into equal parts.

PROPOSITION XXIII. PROBLEM.

To divide a triangle into three equivalent parts by lines drawn from the vertices of the angles to the same point within the triangle.

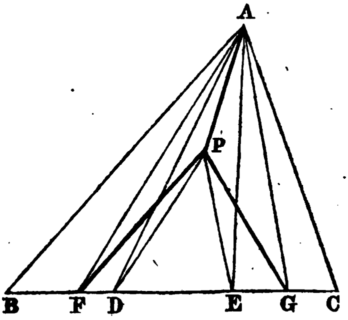


Make BD equal to a third part of BC, and draw DE parallel to BA, the side to which BD is adjacent. From F, the middle of DE, draw the straight lines FA, FB, FC, and they will divide the triangle as required.

For draw DA, then since BD is one third of BC, the triangle ABD is one third of the triangle ABC; but ABD=ABF (Prop. III. Cor. 2. B. II.); therefore ABF is one third of ABC; also since DF=FE, BDF=AFE, likewise CFD=CFE; consequently the whole triangle FBC is equal to the whole triangle FCA; and FBA has been shown to be equal to a third part of the whole triangle ABC; consequently the triangles FBA, FBC, FCA are each equal to a third part of ABC.

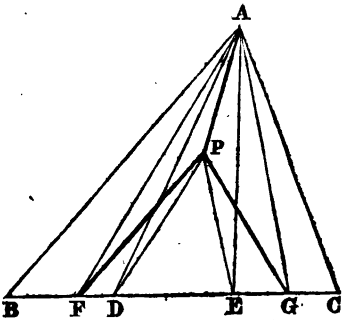
PROPOSITION XXIV. PROBLEM.

To divide a triangle ABC into three equivalent parts by lines drawn from P, a given point within it.



Divide BC into three equal parts in the points D, E, and draw PD, PE; draw also AF parallel to PD, and AG parallel to PE; then if the lines PF, PG, PA be drawn, the triangle ABC will be divided by them into three equivalent parts.

For join AD, AE; then because AF, PD are parallel, the triangle AFP is equivalent to the triangle AFD; consequently, if to each of these there be added the triangle ABF, there will result the quadrilateral ABFP, equivalent to the triangle ABD; but since BD is a third part of BC, the triangle

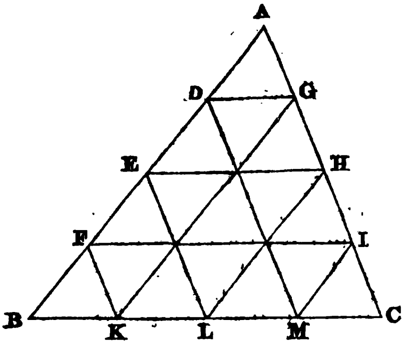


ABD is a third part of the triangle ABC (Prop. I. B. VI.); consequently the quadrilateral ABFP is a third part of the triangle ABC. Again, because AG, PE are parallel, the triangle AGP is equivalent to the triangle AGE; add to each the triangle ACG, and there results the quadrilateral ACGP, equivalent to the triangle ACE; and this triangle is one third of ABC; hence the quadrilateral ACGP is one third of the triangle ABC; consequently, the spaces ABFP, ACPG, FPG are each equal to a third part of the triangle ABC.

PROPOSITION XXV. PROBLEM.

To divide a triangle into any square number of equal triangles, similar to each other and to the original triangle.

Let it be required to divide the triangle ABC into sixteen equal triangles similar to it.

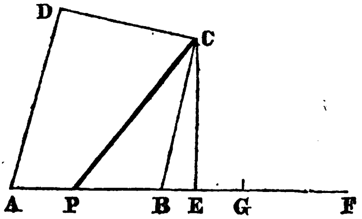


Divide one side AB into four equal parts, and from the the points of division draw DG, EH, FI parallel to BC, and DM, EL, FK parallel to AC. Through the intersections of these parallels draw GK, HL, IM, and the triangle ABC will be divided as required.

For the triangles whose bases are AD, DE, EF, FB, have their sides parallel to those of the triangle ABC, and are therefore similar to it; and as the bases of these triangles are equal, the triangles themselves are equal, and have equal altitudes, so that a single straight line GK passes through the vertices of all, and is parallel to AB. The rhomboid DK is divided into equal rhomboids by the parallels from E and F, and these again are divided into equal triangles by their diagonals. In a similar manner the rhomboid AM is divided into equal triangles, as also the rhomboids DL, EK, BI, and FC, and each of these rhomboids contain one or more of the triangles contained in DK; hence the triangles are all equal, and they are similar to ABC.

PROPOSITION XXVI. PROBLEM.

To divide a quadrilateral into two parts by a straight line drawn from C, the vertex of one of its angles, so that the parts may be to each other as a line M to another line N.



Draw CE perpendicular to AB, and construct a rectangle equivalent to the given quadrilateral, of which one side may be CE; let the other side be EF; and divide EF in G, so that M : N :: GF : EG; take BP equal to twice EG, and join PC, then the quadrilateral will be divided as required.

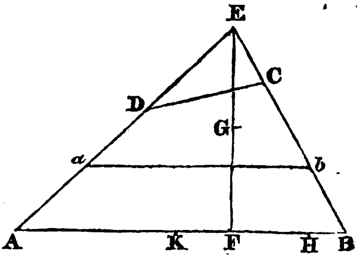
For by construction the triangle CPB is equivalent to the rectangle CE·EG; therefore the rectangle CE·GF is to the triangle CPB as GF is to EG. Now CE·GF is equivalent to the quadrilateral DP, and GF is to EG as M is to N, therefore

DP : CPB :: M : N;

that is, the quadrilateral is divided as required.

PROPOSITION XXVII. PROBLEM.

To divide a quadrilateral AC into two parts by a line parallel to AB one of its sides, so that these parts may be to each other as the line M is to the line N.

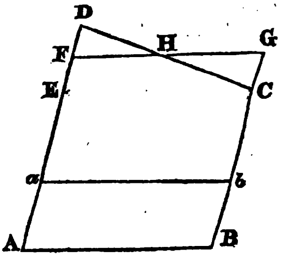


Produce AD, BC till they meet in E; draw the perpendicular EF and bisect it in G. Upon the side GF construct a rectangle equivalent to the triangle EDC, and let HB be equal to the other side of this rectangle. Divide AH in K, so that AK : KH :: M : N, and as AB is to KB, so make DA^2 to Ea^2; draw *ab* parallel to AB, and it will divide the quadrilateral into the required parts.

For since the triangles EAB, E*ab* are similar, we have the proportion $EAB : Eab :: EA^2 : Ea^2$; but by construction, $EA^2 : Ea^2 :: AB : KB$, so that $EAB : Eab :: AB : KB :: AB \cdot GF : KB \cdot GF$; and consequently, since by construction $EAB = AB \cdot GF$, it follows that $Eab = KB \cdot GF$, and therefore $AK \cdot GF = Ab$; and since by construction $AH \cdot GF = AC$, it follows that $KH \cdot GF = aC$. Now $AK \cdot GF : KH \cdot GF :: AK : KH$; but by construction, $AK : KH :: M : N$; consequently,

$$Ab : aC :: M : N;$$

that is, the quadrilateral is divided as required.



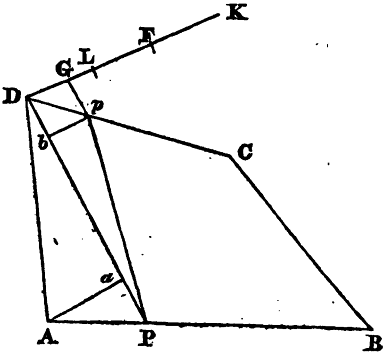
If the sides AD, BC are parallel, then make $AE = BC$, bisect ED in F, and divide AF in *a*, so that $Aa : aF :: M : N$, then the parallel *ab* will divide the quadrilateral as required.

For draw FG parallel to *ab* meeting the production of BC, then since $FD = CG$, the triangles DFH, GCH are equal; so that the quadrilateral *a*C is equivalent to the rhomboid *a*G, and by construction $M : N :: Aa : aF :: Ab : aG$; consequently,

$$M : N :: Ab : aC.$$

PROPOSITION XXVIII. PROBLEM.

To divide a quadrilateral into two parts by a line drawn from P, a point in one of its sides, so that the parts may be to each other as a line M is to a line N.



Draw PD, upon which construct a rectangle equivalent to the given quadrilateral, and let DK be the other side of this rectangle; divide DK in L, so that $DL : LK :: M : N$; make $DF = 2DL$, and FG equal to the perpendicular A*a*; draw G*p* parallel to DP, join the points P, *p*, and the quadrilateral will be divided as required.

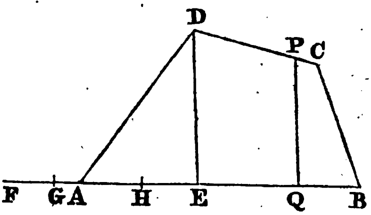
For draw the perpendicular pb:—then, by construction, PD·DK = AC, and PD·DF = PD·Aa + PD.pb, that is, PD·DF is equivalent to twice the sum of the triangles APD, pPD; consequently, since DL is half DF, PD·DL = APpD; and therefore PD·LK = PBCp; but PD·DL : PD·LK :: DL : LK :: M : N; consequently,

$$\mathrm{AP}p\mathrm{D} : \mathrm{PBC}p :: \mathrm{M} : \mathrm{N};$$

hence the quadrilateral is divided as required.

PROPOSITION XXIX. PROBLEM.

To divide a quadrilateral ABCD by a line perpendicular to one of the sides AB, so that the two parts may be to each other as a line M is to a line N.



Construct on DE, perpendicular to AB, a rectangle DE·EF, equivalent to the quadrilateral AC, and divide FE in G, so that FG : GE :: M : N. Bisect AE in H, and (Prop. XXVI.) divide the quadrilateral EC into two parts by a line PQ, parallel to the side DE, so that those parts may be to each other as FG is to GH, then PQ will also divide the quadrilateral AC as required.

For by construction DE·EF = AC, and DE·EH = DAE; hence DE·HF = EC, and consequently, since the quadrilateral EC is divided in the same proportion as the base FH of its equivalent rectangle, it follows that QC = DE·FG, and EP = DE·GH, also AP = DE·GE; consequently,

$$\mathrm{QC} : \mathrm{AP} :: \mathrm{FG} : \mathrm{GE} :: \mathrm{M} : \mathrm{N};$$

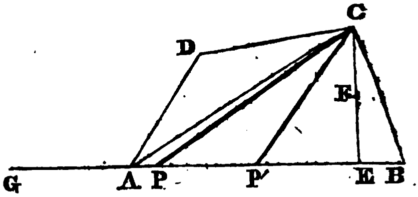
that is, the quadrilateral is divided as required.

PROPOSITION XXX. PROBLEM.

To divide a quadrilateral ABCD into three equivalent parts by lines drawn from C, the vertex of one of its angles.

Draw the diagonal AC, and the perpendicular CE, which bisect in F; construct upon FE a rectangle equivalent to the triangle DAC, and let AG be equal to the other side of this rectangle.

Then the remaining construction of this problem will vary accordingly as AB exceeds twice AG, is less than half AG, or is between these two. Let us first suppose that $AB > 2AG$.



Take AP equal to a third part of $AB - 2AG$; bisect PB in P′; then draw the lines CP, CP′, and they will divide the quadrilateral into three equivalent parts.

For $PB = AB - AP$, and since $AP = \frac{1}{3} AB - \frac{2}{3} AG$, by construction, it follows that $PB = AB - (\frac{1}{3}AB - \frac{2}{3}AG) = \frac{2}{3}AB + \frac{2}{3}AG$, that is to say, $PB = {}^{2} GB$, and therefore $PB \cdot FE = \frac{2}{3} GB \cdot FE$, that is, the triangle CPB is two thirds of the quadrilateral AC; but the triangles CPP′, CP′B having equal bases, PP′, P′B, are each half the triangle CPB, and consequently one third of the quadrilateral: hence the spaces DAPC, CPP′, CP′B are equivalent.

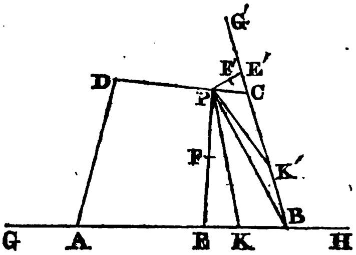
Scholium.

In this case of the problem the lines of division must necessarily fall within the triangle CAB; for AB being greater than 2AG, $AB \cdot FE > 2AG{:}FE$, that is, the triangle CAB exceeds twice the triangle DAC, and is therefore greater than two thirds of the quadrilateral. In the third case of the problem (provided AG is not equal to $\frac{1}{2}$ AB, nor to 2AB) this triangle will be less than two thirds of the quadrilateral, but greater than one third; and consequently one line of division only will fall within the triangle CAB, and the other within the triangle DAC; in this case, therefore, after having determined the line CP′, by making $AP' = \frac{1}{3}(2AB - AG)$, it will only be necessary to divide the remaining quadrilateral AP′CD, by proposition XXVI., into two equal parts, by a line from the point C. In the second case of this problem, that is to say, when AB is less than half AG, the triangle CAB will be less than one third of the proposed quadrilateral, and consequently both lines of division will fall within the other triangle DAC; and therefore this case is virtually the same as the first; the perpendicular from C being to the side AD instead of AB. When we happen to have $AG = \frac{1}{2}$ AB, or $AG = 2AB$, then the diagonal CA will be one line of division.

A mere inspection of the figure will always enable us to determine upon which side the perpendicular is to fall; for as one triangle will be always at least double the other in surface, the greatest will be at once recognized.

PROPOSITION XXXI. PROBLEM.

To divide a quadrilateral AD into three equivalent parts by lines drawn from a given point P in one of its sides.



The construction of this problem will be modified accordingly as the triangle CPB, cut off by the line PB, is less or greater than one third of the quadrilateral AC, which will be discovered by constructing upon a common base rectangles equivalent to the triangle CPB, and to the quadrilateral ADPB:—If the altitude of the former be less than half the altitude of the latter, the triangle CPB will obviously be less than one third of the quadrilateral AC; and if it be greater, then the same triangle will exceed one third of the quadrilateral. Suppose then $CPB < \frac{1}{3} AC$. Draw PE perpendicular to AB, and upon FE, the half of PE, construct a rectangle equivalent to ADPB, and let the other side of this rectangle be equal to GB; construct also on FE a rectangle equivalent to the triangle CPB, and let BH be the other side of this rectangle.

Take $BK = \frac{1}{3} GB - \frac{2}{3} BH$, join PK, then PK will be one of the lines of division.

For the triangle PKB is equal to the rectangle $KB \cdot FE$, and the triangle $CPB = BH \cdot FE$ by construction, therefore, $PKBCP = (KB + BH) \cdot FE$; but by construction, $KB = \frac{1}{3} GB - \frac{2}{3} BH$; therefore $KB + BH = \frac{1}{3} GB + \frac{1}{3} BH = \frac{1}{3} GH$; hence $PKBCP = \frac{1}{3} GH \cdot FE = \frac{1}{3} ABCD$.

If $CPB > \frac{1}{3} AC$, then, instead of the perpendicular PE to AB, draw PE′ perpendicular to BC or its production, and upon F′ E′, the half thereof, construct a rectangle equivalent to ADPB, and let G′B be equal to its base. Take $BK' = \frac{2}{3} BC - \frac{1}{3} BG'$, and join PK′, then PK will be one of the lines of division.

For $K'C = BC - BK'$, and by construction $BK' = \frac{2}{3} BC - \frac{1}{3} BG'$; therefore $K'C = BC - (\frac{2}{3} BC - \frac{1}{3} BG') = \frac{1}{3} BC + \frac{1}{3} BG$, so that the triangle $PK'C = \frac{1}{3} BC \cdot F'E' + \frac{1}{3} BG \cdot F'E'$, which, by construction is equivalent to one third of the whole quadrilateral ABCD.

The other line of division will be readily found by Prop. XXVI.

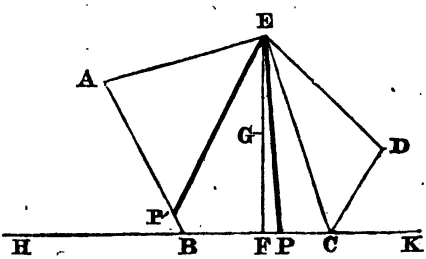
Scholium.

If $GB = \frac{1}{2} BH$, or $BC = \frac{1}{2} BG'$, then the value of BK or BK′ is 0, which must necessarily be the case, since the line PB must then cut off a third part from the quadrilateral AC. This case of the problem will be intimated by the preliminary construction employed as above directed, to ascertain whether the part cut off by this line be greater or less than a third of the quadrilateral. It may be necessary to remark that the common base upon which the rectangles equivalent to these two parts are constituted, may as well be one of the perpendiculars PE or PE′; for should the one chosen happen to be that which is to be drawn in the construction of the problem, it is plain that this construction will thus be forwarded.

PROPOSITION XXXII. PROBLEM.

To divide the irregular pentagon ABCDE into three equivalent surfaces by lines drawn from E, the vertex of one of the angles.

The construction of this problem, like that of the preceding, will vary accordingly as the diagonal EC cuts off a portion EDC, less or greater than the third part of the whole polygon, which may be ascertained by performing the same preliminary operation as was directed in the preceding problem. Suppose that the triangle EDC is less than a third of the polygon, in which case both lines of division must necessarily fall to the left of the diagonal EC.



Draw EF perpendicular to BC, and bisect it in G. Upon GF construct a rectangle $GF \cdot HC$ equivalent to the quadrilateral AC; construct also upon GF a rectangle $GF \cdot CK$ equivalent to the triangle EDC. Take CP

$= \frac{1}{3}$ HC — $\frac{2}{3}$ CK, and draw EP, which will cut off a portion PD equivalent to a third part of the polygon.

For the triangle ECP is equivalent to the rectangle GF·PC, and by construction the quadrilateral AC is equivalent to GF.HC; hence the quadrilateral AP is equivalent to GF(HC — PC), but PC $= \frac{1}{3}$ HC — $\frac{2}{3}$ CK; consequently HC — PC $= \frac{1}{3}$ HC $+ \frac{1}{3}$ CK $= \frac{1}{3}$ HK; therefore AP $= \frac{1}{3}$GF·HK $= \frac{1}{3}$ ABCDE.

Having found one line of division, the other EP′ may be found by proposition XXVI.

If one of the lines of division EP fall to the right of EC, then the perpendicular from E must be drawn to CD, and the remaining construction will suggest itself from what has been done above, and in the preceding problem.

Scholium.

It appears to be unnecessary to extend these problems upon the *division of polygons* to any greater length; a sufficient number has been given on this subject to afford the student an opportunity of applying the principles established in the preceding books to an interesting and useful class of problems. We shall merely add the three following by way of further exercise, and shall terminate this part of our subject with two curious problems relative to the division of the circle.

1. To divide a pentagon by a line drawn from the vertex of one of its angles, so that the parts may be to each other as a line M is to a line N.

2. To divide a pentagon into three equivalent surfaces by lines drawn from a given point in one of its sides.

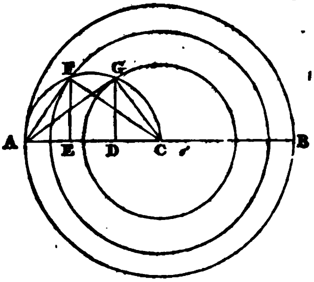
3. To divide a pentagon into three equivalent surfaces by lines drawn from two given points in one of its sides.

PROPOSITION XXXIII. PROBLEM.

To divide a circle into any number of equal parts by means of concentric circles.

Let it be proposed to divide the circle in the margin, whose centre is C, and diameter AB, into a certain number of equal parts, three for instance, by means of circles concentric with it.

Divide the radius AC into three equal parts, AE, ED, DC; draw the perpendiculars EF, DG, meeting the semicircumference described upon AC, in the points F, G: draw CF, CG, and with these lines as radii from the centre C, describe circles: these circles will divide the proposed circle into the required number of equal parts.



For draw AF, AG: then the angle AGC being in a semicircle is a right angle; hence the triangles GAC, GDC are similar, and consequently are to each other as the squares of their homologous sides, that is,

$$\text{GAC} : \text{GDC} :: \text{CA}^2 : \text{CG}^2;$$
$$\text{but GAC} : \text{GDC} :: \text{CA} : \text{CD};$$
$$\text{hence CA}^2 : \text{CG}^2 :: \text{CA} : \text{CD};$$

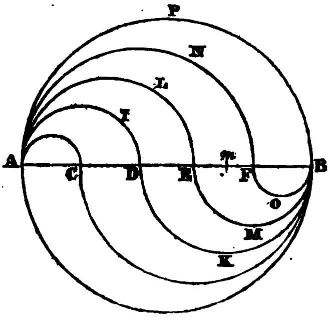
consequently, since circles are to each other as the squares of their radii, it follows that the circle whose radius is CA, is to that whose radius is CG, as CA is to CD, that is to say, the latter is one third of the former.

In like manner, by reasoning on the right angled triangles FAC, FEC, it may be proved that the circle whose radius is CF is two thirds that whose radius is CA. Consequently the smaller circle and the two surrounding *annular* spaces are all equal.

PROPOSITION XXXIV. PROBLEM.

To divide a circle into any number of parts, which shall be all equal both in surface and boundary.

Let it be required to divide the circle whose diameter is AB into five parts, which shall be equal both in surface and boundary. Divide the diameter into five equal parts, in the points C, D, E, F, and upon AC, AD, AE, AF describe semicircles. Describe semicircles also upon BC, BD, BE, BF, but on the opposite side of the diameter AB: then the circle will be divided into the proposed number of curvilinear spaces equal to each other both in surface and boundary.



For the diameter AB is to the diameter AD, as the circumference on AB is to the circumference on AD, or as the semi-circumference on AB is to the semi-circumference on AD: also AB is to BD as the semi-circumference on AB is to that on BD. Consequently AB is to AD and BD together, as the semi-circumference APB is to the boundary AIDKB; therefore these two lines are equal. In a similar manner it may be shown that each of the other boundaries is also equal to the semi-circumference APB.

Again, the circle on AB is to the circles on AE, AF, as the square of AB is to the squares of AE, AF respectively. Consequently,

$$\textit{circ.}\ \mathrm{AB} : \textit{circ.}\ \mathrm{AF} - \textit{circ.}\ \mathrm{AE} :: \mathrm{AB}^2 : \mathrm{AF}^2 - \mathrm{AE}^2.$$

Now (Prop. VII. B. II.) $\mathrm{AF}^2 - \mathrm{AE}^2 = (\mathrm{AE} + \mathrm{AF})\cdot\mathrm{EF}$: Let m be the middle of EF, then $\mathrm{AE} + \mathrm{AF} = 2\mathrm{A}m$. Hence,

$$\textit{circ.}\ \mathrm{AB} : \tfrac{1}{2}\,\textit{circ.}\ \mathrm{AF} - \tfrac{1}{2}\,\textit{circ.}\ \mathrm{AE} :: \mathrm{AB}^2 : \mathrm{A}m\cdot\mathrm{EF};$$

that is to say, the circle on AB is to the space included between the semicircles on AE and AF, as the square of AB is to the rectangle $\mathrm{A}m\cdot\mathrm{EF}$. In exactly the same way it is proved that the circle on AB is to the space between the semicircles on BE and BF, as the square of AB is to the rectangle of $\mathrm{B}m\cdot\mathrm{EF}$. It follows therefore (Prop. XVI. B. V.), that the circle on AB is to the whole space ALEMBOFN, as the square of AB to the sum of the rectangles $\mathrm{A}m\cdot\mathrm{EF}$, $\mathrm{B}m\cdot\mathrm{EF}$, that is, to the rectangle $\mathrm{AB}\cdot\mathrm{EF}$, and this rectangle is one fifth of AB^2; consequently the space ALEMBOFN is one fifth of the circle, and the same may, in like manner, be shown of the other spaces.

NOTES

ON THE

FIRST EIGHT BOOKS OF THE ELEMENTS.

NOTES.

BOOK I.

On the Definitions.

VARIOUS definitions have been proposed by different writers, to distinguish the straight line, but they may all be shown to be liable to some objection; a circumstance not in the least remarkable, for what is meant by a straight line is so generally understood, that it does not seem possible to convey, by any definition, a better notion of it than the mere mention of its name suggests. *Euclid* says, "A straight line is that which lies evenly between its extreme points*;" a definition which is both unsatisfactory and useless. Others, following *Archimedes*, define it as "the shortest distance from one point to another;" but this appears to be assuming too much in a definition, as it immediately leads to the inference, that any two sides of a triangle are together longer than the third side; a proposition which doubtless requires demonstration.

The definition which I have given of straight lines, is, I think, as little liable to objection as can be expected. It has, at least, one advantage: it dispenses with Euclid's tenth axiom, viz. "Two straight lines cannot enclose a space;" a property essential to the demonstration of proposition V. of this book. *Professor Playfair* has defined a straight line as follows: "If two lines are such that they cannot coincide in any two points without coinciding altogether, each of them is called a straight line."

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* This definition, however, as translated by Mr. Playfair, appears less faulty, viz. "A straight line is that which lies *equally* between its extreme points," and in this manner the translation is rendered in the French edition of M. Peyrard.

This definition is not the best that can be given, for it contains more than is requisite. A definition which involves conditions not absolutely necessary is faulty, as these superfluous conditions may be dispensed with, without leaving the thing defined less distinctly characterized. On this account, Euclid's definition of a square, as having "all its sides equal, and all its angles right angles," has been very properly objected to, as containing superfluous conditions: his definition of an isosceles triangle has, on the other hand, been with equal propriety objected to, as being too restricted; since by defining it as "that which has *only* two sides equal," the equilateral triangle is excluded. The meaning of Mr. Playfair's definition is, that if two lines which coincide in any two points are always found to coincide throughout their whole extent, each is a straight line. Now it is only necessary that the lines coincide *between their coinciding points;* for that they will then coincide in every other part may be rigorously demonstrated, as in Prop. V. of these elements. This definition, therefore, is susceptible of restriction. The definition which is given in the text is not liable to this objection; it sufficiently characterizes a straight line, and involves nothing but what must otherwise be assumed as an axiom, viz., that two straight lines cannot include space. Mr. Playfair's definition involves in it, in addition to this, the theorem that "two straight lines cannot have a common segment;" and it is remarkable that that acute geometer should not have perceived that this very circumstance, which in his notes he seems to attribute to the merits of his definition, was in reality a consequence of its defect. I have been thus particular in examining Professor Playfair's definition, because I apprehend that it has hitherto been considered as perfectly unobjectionable, and as possessing the same degree of merit that usually attached to the productions of that distinguished individual.

The definition which Euclid has given of an angle is very vague, and can convey but an indistinct notion of angular magnitude; he calls it "the inclination of two straight lines to one another, which meet together, but are not in the same straight line." To understand this definition, it is necessary previously to know what is meant by "the inclination of two straight lines;" an expression which has not, however, been defined. A modern author of celebrity has endeavoured to give an idea of an angle, by referring to the revolution of a straight line; "A right angle is the fourth part of an entire circuit or revolution of a straight line:" but what an angle has to do with the revolution of a straight line is not easy to conceive; it is certainly not in the smallest degree essential to its existence, for if there were no such thing as a circle, we could quite as readily admit the existence of an angle. The reference of angles to the arcs of a circle is merely an artificial contrivance, adopted for the more convenient measurement and comparison of this class of magnitudes, solely with a view to practical facility; but is in no way connected with the nature of an angle, and is therefore improperly brought forward in its definition. A more usual definition is, "an angle is the opening of two straight lines which meet in a point." By substituting the word *between* for *of*, I think this definition becomes more explicit.

A perpendicular is generally defined as making equal adjacent angles with the line on which it falls; a definition which appears to require amendment, as it excludes the perpendicular at the extremity of a line. It has therefore been thought proper to make the necessary addition.

The definitions of a rhombus, a rectangle, and a square, appear to be rather simpler than those usually given; they involve no more conditions than are absolutely necessary, and those conditions are such as may plainly subsist in the same figure; it being only requisite to admit that one line may be parallel to another, a fact fully established in proposition XII., before either of these definitions are referred to.

It may be proper here to remark, that, in the application of terms, I have, in some instances, ventured to depart from ordinary usage. Thus in the comparison of lines, instead of adopting the customary distinction of *greater* and *less*, I have preferred the designation of *longer* and *shorter*. As a line is understood to be merely length, the terms greater and less appear to be more comprehensive than necessary, and seem to imply other dimensions. For this reason, therefore, they have been changed for terms of more restricted import. I have also confined the term *segment* to the portion of a circle cut off by its chord, although it has been hitherto applied equally to a portion of a straight line. But this extension of the term appears to be quite unnecessary; for as a line has but one dimension, the expressions *part* of a line, or *portion* of a line can convey no ambiguity, and, therefore, on the ground of simplicity, appear to be preferable to the term *segment*. With similar views to precision I have uniformly adopted the term *magnitude* instead of the less definite expression *quantity*. The distinction which *Legendre* has drawn between *equivalent* figures and *equal* figures, I have preserved in this treatise, as also that which subsists between an angle and the vertex of an angle, a distinction not always made; for as he observes, the word *angle* is often employed in ordinary language to designate the point situated at the vertex. These changes and distinctions, trifling as they may appear, are not unimportant, for by giving precision to the terms employed, we avoid at least one cause of obscurity.

On the Theory of Parallel Lines.

The theory of parallel lines is a subject that has considerably perplexed geometers since the time of Euclid. The difficulty consists in showing that if a straight line intersect two parallels it will make the alternate angles equal, a truth which has never yet been established in a manner perfectly unobjectionable and conclusive. Some have attributed this failure to the definition usually given of parallels, and have sought to overcome the difficulty by employing a different definition; still however the same or a similar obstacle has presented itself, and in almost every attempt which has been hitherto made to demonstrate the simplest properties of parallels in a purely geometrical manner, there has been found, upon scrutiny, to lurk some unwarrantable principle tantamount to an assumption of that which it is proposed to demonstrate. Euclid has been charged with having evaded the difficulty, but this seems to be hardly a fair statement of the course which he has adopted. He had no doubt used every effort to overcome it by the aid of previously established principles, but not meeting with success, he found it necessary to assume an additional principle for this express purpose. This principle constitutes his twelfth axiom; as however it is very far from being self-evident, it is entirely misplaced, nor is the principle itself the simplest that could have been chosen. That which forms proposition XIII. in these elements, is doubtless more simple than Euclid's twelfth axiom. This proposition which I have distinguished as a Lemma, forms the ninth axiom in *Mr. Thomas Simpson's* ingenious Elements of Geometry; I have placed it immediately before the proposition to which it is subsidiary, and have endeavoured to establish its truth by a simple reference to one of the most obvious characteristics of a straight line*. The demonstration of the proposition which immediately precedes this Lemma is taken, with some little alteration, from the *Principes Mathématiques, par M. Da Cunha.* This demonstration is superior to every other that has been given of the same proposition. In Euclid, and in most modern authors, this proposition depends upon a subsidiary theorem, which is of no other use whatever (Prop. XVI. Euc.). It is therefore somewhat remarkable, that late writers on geometry have not availed themselves of this decided improvement, instead of following the more circuitous course, which in this instance Euclid has pursued; and more especially as it was so strenuously recommended to their notice by the late learned professor Playfair, who considered it as a most important improvement in elementary geometry. I believe, however, that it has not, till now, found a place in any English book.

As the subject of parallel lines is one of so much interest and importance, it will not be uninstructive to the student to point out to him some of the fallacies which have insinuated themselves into the reasonings of one or two mathematicians of eminence in their writings on this subject, as it will manifest to him the great degree of caution necessary to be observed in these inquiries. The following forms proposition XXII. in the first book of *Leslie's* Elements of Geometry, third edition.

"If a straight line fall upon two parallel straight lines, it will make the alternate angles equal, the exterior angle equal to the interior opposite one, and the two interior angles on the same side together equal to two right angles.

Let the straight line EFG fall upon the parallels AB and CD; the alternate angles AGF and DFG are equal, the exterior angle EFC is equal to the interior angle EGA, and the interior angles CFG and AGF, or FGB and GFD are together equal to two right angles.

For conceive a straight line produced both ways from F, to turn about that point in the same plane; it will first cut the extended line AB above G, and towards A, and will, in its progress, afterwards meet this line on the other side below G, and towards B. In the position IFH, the angle EFH is the exterior angle of the triangle FHG, and therefore greater than FGH or EGA. But in the last position LFK, the exterior angle EFL is equal to its vertical angle GFK in the triangle FKG, and to which the angle FGA is exterior; consequently FGA is greater than EFL, or the angle EFL is less than FGA, or EGA. When the incident line EFG therefore meets AB above the point G, it makes an angle EFH *greater* than EGA, and when it meets AB below that point, it makes an angle EFL, which is *less* than the same angle. But in passing through all the degrees from greater to less, a varying magnitude must evidently encounter the single intermediate limit of equality. Wherefore there is a certain position CD, in which the line revolving about the point F, makes the exterior angle EFC *equal* to the interior EGA, and at the same instant of time meets AB neither towards the one part nor the other, or is parallel to it.

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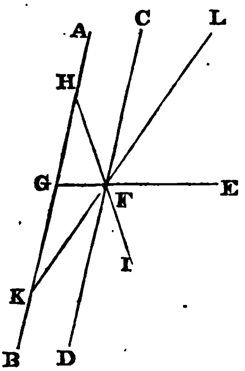
* *M. Le Commandeur de Nieuport* has given the following definition of a straight line, viz., *la ligne droite est celle que parcourt un point A, dirigeant constamment et invariablement sa route vers un même point B.*

Nouveaux Mem. de l'Acad. de Bruxelles, tom. i.



And now, since EFC is proved to be equal to EGA, and is also equal to the vertical angle GFD, the alternate angles FGA and GFD are equal. Again, because GFD and FGA are equal, add the angle FGB to each, and the two angles GFD and FGB are equal to FGA and FGB, but the angles FGA and FGB on the same side of AB are equal to two right angles, and consequently the interior angles GFD and FGB are likewise equal to two right angles."

The above demonstration is unfortunately an entire failure; its fallacy was first pointed out in the *Edinburgh Review*, in a critique on the second edition of Mr. Leslie's work; as however the demonstration remains unaltered in the third edition, it must be inferred that the learned author continues satisfied with the accuracy of the reasoning which he has employed. A little examination, however, appears sufficient to discover that Professor Leslie has in reality been demonstrating, not the proposition enunciated, but the converse of it, viz. If a straight line fall on two others and make the alternate angles equal, the two lines will be parallel. For he shows that there is a certain position CD, in which the revolving line makes the exterior angle EFC equal to the interior EGA, and that then it must be parallel to AB; but it is not shown that these lines can never be parallel but in this particular position, which is the only thing difficult to prove, and which indeed it was the object of the proposition to demonstrate.

Another attempt to establish this theory has been more recently made by the writer of the article "Geometry," in the *Encyclopædia Metropolitana*, a work of high value. The writer sets out with an entirely new definition of parallels, viz., "Parallel lines are those in which any point being taken in the one, and any point being taken in the other, the perpendicular distance of these points from the other line shall be equal to each other." By means of this definition the usual theorems are readily deduced. But these theorems are by no means sufficient to complete the theory of parallels according to the above new definition. It is further necessary to prove, that straight lines which are not parallel must necessarily meet: a proposition which is not established in the treatise referred to, nor can it indeed be established without the aid of some such assumption as preceding writers have been compelled to make, in order to get through the difficulties of this subject. Whatever definition be given of parallel lines, it is absolutely indispensable to establish this distinction between those which are parallel, and those which are not, viz., that the former can never meet when produced, but that the latter necessarily must: a distinction implied in Euclid's definition of parallels, and without which it would be quite impossible to show that any two lines whatever, however they may be drawn in reference to a third, can be produced till they meet. Hence the construction of problem VIII. in the article alluded to, viz., "Given two angles and any side of a triangle to construct the triangle" is not substantiated, for two lines are assumed to meet, when, for aught that has been shown to the contrary, they may be incapable of meeting. I am aware that to proposition XXIV. Book II. of the aforesaid treatise, where it is demonstrated that *The three angles of a triangle taken together are equal to two right angles*, there is subjoined the following corollary, viz., "It follows from this, that if two lines are cut by a third line, so as to make the two interior angles on the same side less than two right angles, these lines produced will meet, and form a triangle;" from a slight examination, however, it will appear that this corollary does *not* follow.

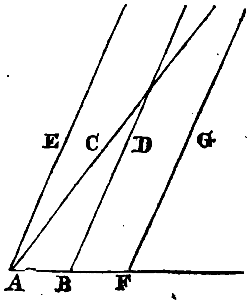
A very novel mode of considering this subject has been proposed by *M. Bertrand* of Geneva, which has attracted much notice on the continent; *M. Develey* has introduced it among the propositions of his *Elémens de Géométrie*, and *Lacroix* has given it in a note at page 23 of his *Elémens*, and has pronounced it to be more simple and ingenious than any with which he is acquainted *; although I cannot help regarding it as a mere contrivance, ingenious indeed, but involving considerations that ought not to be admitted into elements of geometry. The reasoning of M. Bertrand to show that two lines which make with a third two interior angles, whose sum is less than two right angles, may be produced till they meet, is in substance as follows:—

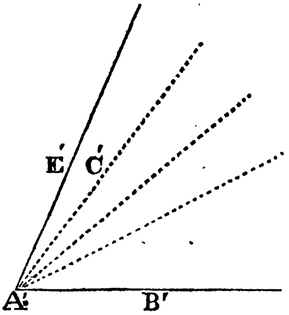
Let the straight lines CA, DB make, with AB, interior angles at A and B, whose sum is less than two right angles, then AC, BD may be produced till they meet.

Let the angle EAB make, with the angle DBA, two right angles, then AC must lie within the sides of the angle EAB. Take the angle E′A′B′ equal to EAB, and E′A′C′ equal to EAC.

Then it is plain, that whatever be the magnitude of the angle E′A′C′, a multiple of it may be taken so great as to exceed the angle E′A′B′; in other words, the angle E′A′C′ may be repeated about the vertex A′ till it fills up the angle E′A′B′; and, consequently, the unlimited space comprised between the lines A′E′, A′C′ will, by such repetition, entirely fill up the unlimited space comprised between the lines A′E′, A′B′, however far the lines comprising these spaces be produced; therefore the space EAC will, by repetition, fill up the space EAB.

Let us now consider the unlimited space, or band EABD, which we may repeat as often as we please upon the production of the base AB; for if BF be taken equal to AB, and the angle BFG be made





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* "Elle m'a paru le plus simple et la plus ingenieuse de toutes celles que je connais."—*Lacroix, Eléméns de Géométrie*, p. 23.

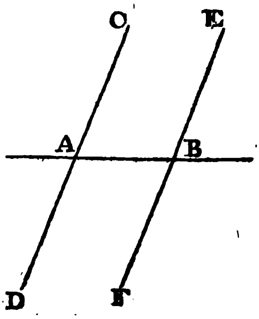
equal to ABD, it is obvious that the band EABD will, upon application, entirely coincide with the band DBFG, for the angles at their bases are equal each to each, and the bases themselves are equal. We may thus, therefore, multiply these bands to any extent by producing AB indefinitely; and yet we shall never be able entirely to fill up the unlimited space comprised between the lines AE, AB. But it has been shown that the space comprised between AE, AC, by being repeated a limited number of times, will entirely fill up the same space, viz., that comprised between AE, AB. It follows, therefore, that the space comprised between AE, AC must exceed the space EABD, and, therefore, cannot possibly be wholly included in that space, which must, however, be the case if AC were not to meet BD; AC, therefore, must necessarily meet BD.

Such is the contrivance which *M. Bertrand* adopts to demonstrate the truth of Euclid's twelfth axiom. His reasoning has, indeed, the semblance of geometrical accuracy, but the consideration of *unlimited* or *infinite spaces*, which that reasoning involves, is not among those comprehended by elementary geometry, which treats only of *figure* or *bounded space;* and, therefore, the validity of the conclusions derived from such considerations may justly be disputed.

The same author has given a neat and a much more satisfactory demonstration of the converse of this proposition; and although I think it is inferior to that of *M. Da Cunha,* it nevertheless deserves notice.

Let the lines CD, EF make, with the line AB, angles CAB, EBA, which are together equal to two right angles; CD, EF are parallel.

Since the angles CAB, DAB are also equal to two right angles, it follows that the angle DAB is equal to the angle EBA.



For similar reasons the angle FBA is equal to the angle CAB. Let DABF be applied to EBAC, so that the angle DAB may coincide with the equal angle EBA, and the angle FBA with the equal angle CAB. Then, since the position of AB remains unaltered, it is obvious that the lines AD, BF will coincide respectively with BE, AC. Hence, if BE, AC could meet, AD, BF would also meet, so that the lines CD, EF would have two points in common, which is impossible. In the same way it would evidently result that if AD, BF could meet, BE, AC would also meet; consequently the lines CD, EF are parallel.

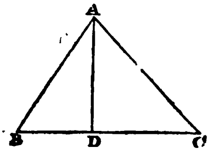
Having now exhibited to the student some of the latest attempts to establish the theory of parallel lines upon geometrical principles, and having endeavoured to point out wherein these attempts fail. I shall now proceed to give an explanation of the method which the celebrated *Legendre* has employed to accomplish the same object, and in which he appears to have entirely succeeded, not indeed in a manner strictly geometrical, but by the aid of principles as simple and as admissible as the axioms of geometry. In extracting from Legendre I shall avail myself of the English translation of *Dr. Brewster.*

Referring to Euclid's twelfth axiom, Legendre observes, "This postulate has never hitherto been demonstrated in a way strictly geometrical, and independent of all considerations about infinity; a circumstance attributable, doubtless, to the imperfection of our common definition of a straight line, on which the whole of geometry hinges." But viewing the matter in a more abstract light, we are furnished by analysis with a very simple method of rigorously proving both this and the other fundamental propositions of geometry. We here propose to explain this method, with all requisite minuteness, beginning with the theorem concerning the sum of the three angles of a triangle.

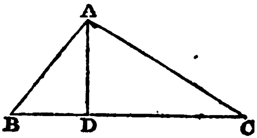
By superposition it can be shown immediately, and without any preliminary propositions, that *two triangles are equal when they have two angles and an interjacent side in each equal.* Let us call this side p, the two adjacent angles A and B, the third angle C. This third angle C, therefore, is entirely determined, when the angles A and B, with the side p are known; for if several different angles C might correspond to the three given magnitudes A, B, p, there would be several different triangles, each having two angles, and the interjacent side equal, which is impossible; hence the angle C must be a determinate function of the three quantities A, B, p, which I shall express thus, $C = \varphi : (A, B, p)$.

Let the right angle be equal to unity, then the angles A, B, C will be numbers included between 0 and 2; or taking two right angles for unity, which indeed seems preferable, since this is the *natural* limit of angular magnitude, all angles will be included between 0 and 1; and since $C = \varphi : (A, B, p)$, I assert, that the line p cannot enter into the function φ. For we have already seen that C must be entirely determined by the given quantities A, B, p alone, without any other line or angle whatever. But the line p is heterogeneous with the numbers A, B, C; and if there existed any equation between A, B, C, p, the value of p might be found from it in terms of A, B, C; whence it would follow that p is equal to a number, which is absurd: hence, p cannot enter into the function φ, and we have simply $C=\varphi:(A, B)$.

This formula already proves that if two angles of one triangle are equal to two angles of another, the third angle of the former must also be equal to the third angle of the latter; and this granted, it is easy to arrive at the theorem we have in view.



First, let ABC be a triangle right angled at A; from the point A draw AD perpendicular to the hypotenuse. The angles B and D of the triangle ABD are equal to the angles B and A of the triangle BAC; hence, from what has been just proved, the third angle BAD is equal to the third C. For a like reason the angle $DAC=B$; hence $BAD+DAC$, or $BAC=B+C$, but the angle BAC is right: hence *the two acute angles of a right angled triangle are together equal to a right angle.*



Now, let BAC be any triangle, and BC a side of it not less than either of the other sides; if from the opposite angle A the perpendicular AD is let fall on BC, this perpendicular will fall within the triangle ABC, and divide it into two right angled triangles BAD, DAC. But, in the right angled triangle BAD, the two angles BAD, ABD are together equal to a right angle; in the right angled triangle DAC, the two DAC, ACD are also equal to a right angle; hence, all four taken together, or, which amounts to the same thing, all the three, BAC, ABC, ACB, are together equal to two right angles; hence, *in every triangle, the sum of its three angles is equal to two right angles.*"

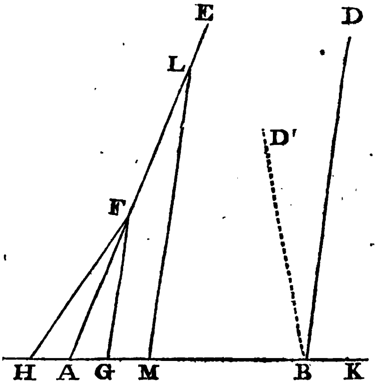
This theorem is sufficient to remove the difficulty in the theory of parallels, as will be afterwards shown. But let us proceed with Legendre's reasoning.

"It thus appears that the theorem in question does not depend, when considered *a priori*, upon any series of propositions, but may be deduced immediately from the principle of homogeneity; a principle which must display itself in all relations between all quantities of whatever sort. Let us continue the investigation, and show that from the same source the other fundamental theorems of geometry may likewise be derived.

Retaining the same denominations as before, let us farther call the side opposite the angle A by the name of m, and the side opposite B by that of n. The quantity m must be entirely determined by the quantities A, B, p alone; hence m is a function of A, B, p, and $\frac{m}{p}$ is one also; so that we may put $\frac{m}{p}=\psi:(A, B, p)$. But $\frac{m}{p}$ is a number as well as A and B; hence the function ψ cannot contain the line p, and we shall have simply $\frac{m}{p}=\psi:(A, B)$, or $m=p\psi:(A, B)$. Hence, also, in like manner, $n=\psi:(B, A)$.

Now, let another triangle be formed with the same angles A, B, C, and with sides m' n' p' respectively opposite to them. Since A and B are not changed, we shall still, in this new triangle, have $m'=p'\psi:(A, B)$, and $n'=p'\psi:(A, B)$. Hence $m:m'::n:n'::p:p'$. Hence, *in equiangular triangles, the sides opposite the equal angles are proportional.*"

From this general proposition Legendre is enabled to demonstrate Euclid's twelfth axiom, viz.: If a straight line meet two straight lines so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced shall at length meet upon that side on which are the angles which are less than two right angles.



Let the straight line AB meet the two straight lines AE, BD, making the interior angles EAB, DBA, together, less than two right angles. It is obvious that at least one of these angles must be acute, the other may be either right, obtuse, or acute. Let EAB be an acute angle, and suppose, first, that the angle DBA is either right or obtuse. Take any point F in AE, and draw FG, making the angle FGB equal to the angle DBK. The point G cannot fall on A, for the angle FAB is less than the angle DBK. Nor can it fall on H in the production of BA; for, since the two angles EAB, DBA are, together, less than two right angles, FAH, DBK must be, together, greater than two right angles; so that if G fell on H, the angles of the triangle FHA would be, together, greater than two right angles, which is impossible; hence G must fall as the figure represents in the direction AB.

Let, now, AL be taken double of AF, and let LM be drawn; making the same angle with AB as FG does. Then, since the triangles FAG, LAM have the angles FGA, LMA equal, while the angle FAG is common to both, they are equiangular: hence we have the proportion AF : AL : : AG : AM; AM, therefore, is double of AG. In a similar manner, if AL had been taken equal to any other multiple of AG, AM would have been an equimultiple of AG; and since some multiple of AG may exceed AB, that is, since the point M may fall beyond B in the production of AB, AE produced must intersect the production of BD, for LM must be always parallel to DB (Prop. XII. Cor. 1. B. I.); and, consequently, when M is beyond B, ML must lie throughout beyond BD, and, therefore, AL must intersect BD.

If AE, BD′ both make acute angles with AB, then it is obvious, since, as just proved, AE may be produced till it meet a perpendicular to AB, from B it must necessarily intersect the intermediate line BD′.

Thus, then, by the aid of a few simple principles derived from the consideration of functions, Euclid's famous postulate becomes susceptible of complete demonstration. Legendre remarks, " the proposition concerning the square of the hypotenuse, we already know is a consequence of that concerning equiangular triangles. Here, then, are three fundamental propositions of geometry—that concerning the three angles of a triangle, that concerning equiangular triangles, and that concerning the square of the hypotenuse, which may be very simply and directly deduced from the consideration of functions."

The above theory of M. Legendre has met with much opposition in this country. It has been said that if the functional equation $C = \varphi : (A, B, p)$ lead to the conclusion that the angle C is simply a function of the other two angles A, B, because " the line p is of a nature heterogeneous to the angles A and B, and, therefore, cannot be compounded with these quantities *;" there would be the same reason to infer from the equation $c = \phi : (a, b, C)$, in which a, b are the sides, and C the included angle of a triangle, whose base is c; that "c is simply a function of a and b, or it is the necessary result merely of the other two sides. In other words, as the third angle of a triangle depends on the other two angles, so the base of a triangle must have its magnitude determined by the lengths of the two incumbent sides. Such is the extreme absurdity to which this sort of reasoning would lead †."

But, in order to overcome whatever doubt the student may entertain of the validity of the reasoning here impugned, I shall venture to offer a few illustrative remarks with a view of placing the subject in a clearer light.

And, first, it may be observed that angles and lines are magnitudes in their nature totally distinct, and unsusceptible, therefore, of comparison. The magnitude of an angle is entirely independent of the length of its including lines, as these are independent of their included angle. An angle, moreover, differs, as magnitude from a line in another respect, viz., it is *naturally* expressible numerically, that is to say, independently of any assumed standard as the measuring unit. With respect to lines, on the contrary, we are necessarily constrained to adopt some arbitrary length as a standard of measurement, such as an inch, a foot, a yard, &c.; there being no definite and invariable standard naturally suggested from the consideration of a straight line, for, strictly speaking, there is no such thing as a whole or complete straight line, or one incapable of further increase or extension; were this, indeed, the case, the adoption of an arbitrary unit of measure would be quite useless and absurd, since straight lines in general would be naturally expressed by numbers denoting them as parts of this whole, or as expressing their ratios thereto; but the straight line has no limit. The limit of an angle on the other hand is two right angles; this is its *maximum* of value, so that every angle is a definite part of this finite whole ‡. Two right angles, therefore, being 1, every angle is expressed by an abstract number lying between the limits of 0 and 1.

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* This quotation needs correction: it is not sufficient that p be heterogeneous to the two magnitudes A, B, but it must be likewise heterogeneous to the third magnitude C.

† *Leslie's Geometry*, third edition, page 294.

‡ We may add, however, that if we chose to ascend to a more elevated analytical principle, it were easy to show that between angles and *one* straight line there cannot

Although straight lines cannot for a moment be conceived to admit of such numerical representation in the absence, be it remembered, of a *conventional* standard of reference; yet when any two or more straight lines are concerned in any inquiry, an abstract number may be the result of their combination; for the ratio* of any two homogeneous quantities whatever must be an abstract number, whether the quantities themselves are determinable numerically or not.

It appears, then, that in our calculations with angles and straight lines, if we previously lay down the principle that *all artificial aids are to be rejected, and the magnitudes in question to be considered as they naturally present themselves to our examination;* angles must be regarded as ratios or abstract numbers, and lines simply as such having no natural standard of comparison.

If, in the equation $C=\phi:(A, B, c)$, c did not disappear, an absolute length, c, might be determined by numbers without the unit of length being known; which is absurd †. But independently of this consideration, it is, I think, quite obvious from the preceding remarks, that whatever be the operation indicated by ϕ upon the numbers A, B and the line c, the result must necessarily be a line; and, therefore, if c did not disappear, the equation $C=\phi:(A, B, c)$ would be impossible, although at the commencement of the reasoning it was admitted to exist.

In the equation $c=\phi:(a, b, C)$, however, we have obviously no right to expunge either a, b, or C; for there is no absurdity in allowing a straight line c to result from the combination of two other straight lines with a number.

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exist any relation from which the latter as a function of the former can be determined. These quantities, considered as *magnitude* destined to enter into our calculations, are not homogeneous when referred to the wholes, of which they respectively form a part. The angle is a portion of a finite whole, the straight line a portion of an infinite whole; so that *every angle is a finite quantity*, whilst *every given straight line is a quantity infinitely small*, and *only the ratios of given straight lines can enter into our calculations with given angles.—Defence of Legendre's Theory by M. Le Baron Maurice; Legendre's Geometry by Brewster*, p. 235.

* Ratio is defined at page 197.

† "Car si c ne disparoit pas, il faudra qu'une longueur absolue, c, soit determinée par des nombres sans que l'unité de longueur soit connue; ce qui est une absurdité."—*M. Legendre's Letter to Mr. Leslie.*

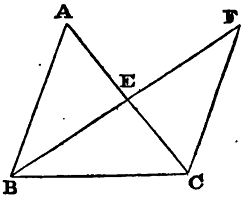
Hence it would be wrong to conclude that the base of a triangle is simply a function of the other two sides, although we cannot avoid inferring that any angle of a triangle is simply a function of the other two angles *.

This rejection of artificial assistance adds much to the elegance of M. Legendre's method, and renders its alliance with pure geometry, which is altogether independent of this sort of aid, the more striking and remarkable.

Having thus explained the analytical method of Legendre, I shall now proceed to lay before the student a very ingenious and satisfactory train of argument, by which the same conclusions are established upon strictly geometrical principles. It is the production of a very distinguished mathematician †; and although it must be confessed that it is not sufficiently simple to be introduced into the first book of the elements; it, nevertheless, clearly shows that the difficulty is not of such a nature as to be beyond the power of elementary geometry to remove.

PROPOSITION I.

To construct a triangle that shall have the sum of its angles equal to the sum of the angles of a given triangle, and one of its angles equal to, or less than, half any proposed angle of the given triangle.



Let ABC be the given triangle, and ABC one of its angles; bisect the side AC, opposite to ABC, in E; join BE, and, having produced it, cut off EF equal to BE; join CF: the sum of the angles of the triangle BFC will be equal to the sum of the angles of the triangle ABC; and one of the angles FBC, or BFC, will be equal to, or less than, half the angle ABC.

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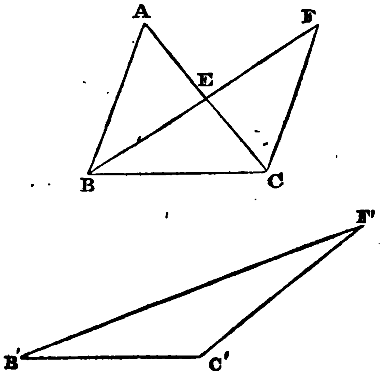
* In addition to the above remarks, the student may consult the paper of M. Le Baron Maurice, which forms part of Note II., in Dr. Brewster's translation of Legendre's Elements of Geometry. See also Philosophical Magazine, vols. 63 and 65.

† James Ivory, Esq. M.A., F.R.S., &c.

The two triangles AEB, CEF are equal; for the two sides AE, EB, and the included angle in the one, are respectively equal to the two sides CE, EF, and the included angle in the other. Wherefore the angle BAE being equal to ECF, the whole angle BCF is equal to the two angles BAE and BCE; and the angle ABE being equal to EFC, the whole angle ABC is equal to the two angles CBE and EFC. Consequently the three angles BCF, CBE, and EFC are equal to the three angles BAC, ACB, and ABC. Again, if BC be equal to CF, the angles EBC and EFC will be equal to one another, and to half of ABC; but, if BC and CF be unequal, the angles EBC and EFC will likewise be unequal; and, therefore, one of them will be less than the half of ABC.

PROPOSITION II.

The three angles of a triangle cannot be greater than two right angles.

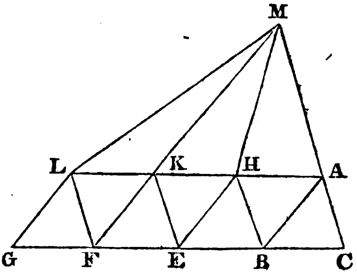


If it be possible, let the three angles of the triangle ABC be greater than two right angles, and let the excess above two right angles be equal to the angle x. Construct the triangle BCF, having the sum of its angles equal to the sum of the angles of the triangle ABC, and one angle FBC equal to, or less than, half the angle ABC; in like manner construct another triangle F′B′C′, having the sum of its angles equal to the sum of the angles of the triangle FBC, and one angle F′B′C′ equal to, or less than, half the angle FBC; and continue the like constructions as far as necessary. Because the angle FBC is equal to, or less than, half the angle ABC; and the angle F′B′C′ equal to, or less than, half the angle FBC, and so on; by continuing the series of triangles far enough, we shall at length arrive at one, viz., F′B′C′, having an angle F′B′C′ less than the given angle x*. And because the three angles of every triangle in the series make the same sum, the three angles B′C′F′, B′F′C′, F′B′C′ will be, together, equal to the sum of two right angles, and the angle x; wherefore the angles B′C′F′ and B′F′C′ are greater than two right angles, which is absurd (Euc. Prop. XVII. B. I.). Therefore the three angles of a triangle cannot be greater than two right angles.

PROPOSITION III.

The three angles of any triangle are equal to two right angles.

If what is affirmed be not true, let the three angles of the triangle ACB be less than two right angles, and let the defect from two right angles be equal to the angle x. Let P stand for a right angle, and find a multiple of the angle x, viz., $m \times x$, such that $4P - m \times x$, or the excess of four right angles above the multiple angle shall be less than the sum of the two angles ACB and ABC of the proposed triangle.



Produce the side CB, and cut off BE, EF, FG, &c. each equal to BC, so that the whole CG shall contain CB m times; and construct the triangles BHE, EKF, FLG, &c., having their sides equal to the sides of the triangle ACB, and, consequently, their angles equal to the angles of the same triangle. In CA produced take any point M, and draw HM, KM, LM, &c.; AH, HK, KL, &c.

All the angles of all the triangles into which the quadrilateral figure CGLM is divided, constitute the four angles of that figure, together with the angles round each of the points H, K, &c., and the angles directed into the interior of the figure, at the points A, B, E, F, &c. But all the angles round the points H, K, &c., of which points the number is $m-2$, are equal to $(m-2) \times 4P$, or to $4mP - 8P$; and all the angles at the points A, B, E, F, &c., are equal to $m \times 2P$. Wherefore the sum of all the angles of all the triangles into which the quadrilateral CGLM is divided, is equal to the four angles of that figure, together with $4mP - 8P + 2mP = 6mP - 8P$.

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* For, by continually bisecting any proposed magnitude, a magnitude will at length be found less than any given magnitude.

Again: the three angles of the triangle ABC are, by hypothesis, equal to $2P - x$; and, as the number of the triangles CAB, BHE, EKF, FLG, is equal to m, the sum of all the angles of all these triangles will be equal to $2mP - m \times x$. Upon each of the lines AH, HK, KL, there stand two triangles, one above, and one below; and, as the three angles of a triangle cannot exceed two right angles, it follows that all the angles of those triangles, the number of which is equal to $2m - 2$, cannot exceed $4mP - 4P$. Wherefore the sum of all the angles of all the triangles into which the quadrilateral CGLM is divided cannot exceed $4mP - 4P + 2mP - m \times x = 6mP - 8P + 4P - m \times x$.

If follows from what has now been proved, that the four angles of the quadrilateral CGLM, together with $6mP - 8P$, cannot exceed $6mP - 8P + 4P - m \times x$. Wherefore, by taking the same thing, viz., $6mP - 8P$, from the two unequal things, the four angles of the quadrilateral CGLM cannot exceed $4P - m \times x$. But $4P - m \times x$ is less than the sum of the two angles ACB and LGF: wherefore *a fortiori*, the four angles of the quadrilateral cannot exceed the sum of the two angles ACB, LGF; that is, a whole cannot exceed a part of it, which is absurd. Therefore the three angles of the triangle ABC cannot be less than two right angles.

And because the three angles of a triangle can neither be greater nor less than two right angles, they are equal to two right angles.

By help of this proposition, observes Mr. Ivory, the defect in Euclid's Theory of Parallel Lines may be removed.

I shall, however, venture to suggest a trifling improvement, which the above reasoning appears to admit of, and thereby obviate an objection that might be brought against it.

It might be said, and with reason, that we have no right to assume, that, in every case, a multiple of x may be taken, such that $4P - mx$ may be less than the sum of the two angles ACB and ABC; for these angles may be so small that their sum shall be much less than the angle x, however small this be assumed; and although $4P - mx$ must also be less than x, it may nevertheless be comparatively much greater than the sum of the angles ACB, ABC; in which case the above conclusion cannot be drawn.

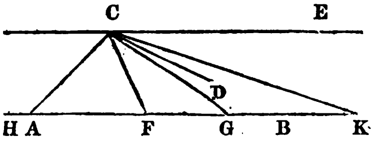
It appears, therefore, preferable to assume the multiple of x, such that mx may *exceed* 4P, which is unquestionably allowable: then the subsequent reasoning may remain the same till we come to the inference, that the four angles of the quadrilateral, together with $6m\mathrm{P}-8\mathrm{P}$, cannot exceed $6m\mathrm{P}-8\mathrm{P}+4\mathrm{P}-mx$, which obviously involves an absurdity, because $6m\mathrm{P}-8\mathrm{P}$ alone exceeds $6m\mathrm{P}-8\mathrm{P}+4\mathrm{P}-mx$; since this latter expression results from adding to the former a *less* magnitude, viz., 4P, and taking away a *greater*, viz., mx, for by hypothesis $4\mathrm{P}>mx$.

For the sake of the young student, it will not be superfluous now to show how the theory of parallels may be rigorously established by the help of the theorem, that the three angles of any triangle amount to two right angles: for this purpose the two propositions following are given in the notes to Playfair's Geometry.

PROPOSITION I.

Two lines, which make with a third line the interior angles on the same side of it less than two right angles, will meet on that side, if produced far enough.

Let the straight lines AB, CD, make with AC, the two angles BAC, DCA less than two right angles; AB and CD will meet if produced toward B and D.



In AB take AF=AC; join CF; produce BA to H, and through C draw CE, making the angle ACE equal to the angle CAH.

Because AC is equal to AF, the angles AFC, ACF, are also equal; but the exterior angle HAC is equal to the two interior and opposite angles ACF, AFC, and therefore it is double of either of them, as of ACF.

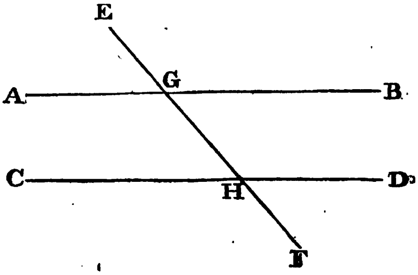
Now, ACE is equal to HAC by construction, therefore ACE is double of ACF, and is bisected by the line CF.

In the same manner, if FG be taken equal to FC, and if CG be drawn, it may be shown that CG bisects the angle FCE, and so on continually. But if from a magnitude, as the angle ACE, there be taken its half, and from the remainder FCE its half FCG, and from the remainder GCE its half, &c., a remainder will at length be found less than the given angle DCE. Let GCE be the angle whose half ECK is less than DCE, then a straight line CK is found, which falls between CD and CE, but nevertheless meets the line AB in K; therefore CD, if produced, must meet AB in a point between G and K, therefore, &c.

PROPOSITION II. (29. I. Euclid.)

If a straight line fall on two parallel straight lines it makes the alternate angles equal to one another; the exterior equal to the interior and opposite on the same side; and likewise the two interior angles on the same side equal to two right angles.

Let the straight line EF fall on the parallel straight lines AB, CD; the alternate angles AGH, GHD are equal; the exterior angle EGB is equal to the interior and opposite angle GHD, and the two interior angles BGH, GHD are equal to two right angles.



For if AGH be not equal to GHD, let it be greater; then adding BGH to both, the angles AGH, HGB are greater than the angles DHG, HGB. But AGH, HGB are equal to two right angles; therefore BGH, GHD are less than two right angles, and therefore the lines AB, CD will meet, by the last proposition, if produced toward B and D.

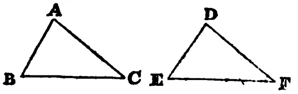
But they do not meet, for they are parallel by hypothesis, and therefore the angles AGH, GHD are not unequal; that is, they are equal to one another.

Now, the angle AGH is equal to EGB, because these are vertical; and it has been also shown to be equal to GHD, therefore EGB and GHD are equal. Lastly, to each of the equal angles EGB, GHD add the angle BGH, then the two EGB, BGH are equal to the two DHG, BGH. But EGB, BGH are equal to two right angles, therefore BGH, GHD are also equal to two right angles, therefore, &c.

On the superposition of figures.

Throughout the whole of this first book, I have endeavoured to avoid as much as possible that method of proof, by which the equality of figures is inferred from the principle of superposition, or the laying of one figure upon another, a mode of proceeding too often resorted to by modern geometers. Mr. Thomas Simpson was particularly averse to this kind of evidence; so much so, that rather than place the fundamental proposition relating to the equality of triangles, where this method of proof appeared to be unavoidable, among his theorems, he preferred to consider it as an axiom, observing in a note that "what is here laid down as an axiom would more properly have been made a proposition, had it admitted of such a demonstration as is perfectly consistent with geometrical strictness and purity. But the laying of one figure upon another, whatever evidence it may afford, is a mechanical consideration, and depends on no postulate." The proposition here alluded to forms proposition VIII. in these elements, and the equality of the triangles in question is there proved by conceiving a third triangle, equal to the one, to be constructed upon a side of the other; although this is, perhaps, virtually the same as laying one triangle upon the other, yet, as it does not suppose the actual operation to be performed, it appears to be less liable to objection. But there are one or two other propositions in this first book relating to the equality of triangles, where the principle of superposition has been quite unnecessarily introduced by modern authors. For instance, proposition XI. is demonstrated by Legendre as follows:—

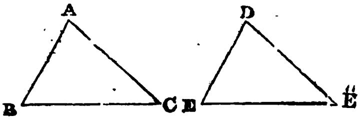
Let the side BC be equal to the side EF, the angle B to the angle E, and the angle C to the angle F; then will the triangle DEF be equal to the triangle ABC.



For, to bring about the superposition, let EF be placed on its equal BC; the point E will fall on B, and the point F on C. And, since the angle E is equal to the angle B, the side ED will take the direction BA; therefore the point D will be found somewhere in the line BA. In like manner, since the angle F is equal to the angle C, the line FD will take the direction CA, and the point D will be found somewhere in the line CA. Hence the point D occurring at the same time in the two straight lines BA and CA must fall on their intersection A; hence the two triangles ABC, DEF coincide with each other, and are perfectly equal.

This demonstration is not shorter than that given in these elements, which is similar to Euclid's, where the truth of the theorem is in no way dependent upon superposition; and although the evidence may not be stronger in the one case than in the other, yet it is doubtless improper to bring forward even an apparently objectionable principle, when the circumstances of the case do not require its aid. In some instances, however, the demonstration may be shortened by adverting to superposition; although, I think, that in general what is gained in this way is lost in point of elegance; for his reason the demonstration of proposition XXVI. of this book has been preferred to the following shorter demonstration, in which superposition is employed.

In the triangles ABC, DEF, let the side AB be equal to DE, BC to EF, and the angles BAC, EDF, opposite to BC, EF, be also equal; the triangles themselves are equal, if the other angles BCA, and EFD opposite to BC, EF be of the same character, or at once right, or acute or obtuse.



For the triangle ABC being applied to DEF, the angle BAC will adapt itself to EDF, since they are equal; and the point B must coincide with E, because the side AB is equal to DE. But the other equal sides BC and EF, now stretching from the same point E towards DF, must likewise coincide; for if the angle at C or F be right, there can exist no more than one perpendicular EF; and, in like manner, if this angle at F be either obtuse or acute, the line EF which forms it, can, for the same reason, have only one corresponding position.—Whence, in each of these three cases, the triangle ABC admits of a perfect adaptation with DEF*.

On Proposition XXIII.

The demonstration of this proposition is rendered much more simple and concise than the demonstrations usually given. That of Euclid, indeed, appears as simple, but as Mr. Thomas Simpson has justly observed in the notes to his geometry, this demonstration is defective; for Euclid has not shown that the extremity H, of the line AH, must necessarily fall below the line BC, but has improperly assumed this to be the case. To supply this defect Euclid's demonstration would require considerable modification. Legendre divides the proposition into three cases, and demonstrates each separately; Professor Leslie distinguishes two cases. The demonstration in these elements is comprehended in a single case, and is as concise as Euclid's without its defect.

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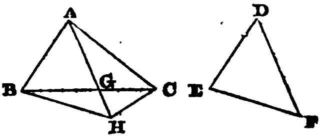
* *Leslie's Geometry*, proposition XXI. book I.



BOOK II.

On Propositions VIII. and IX.

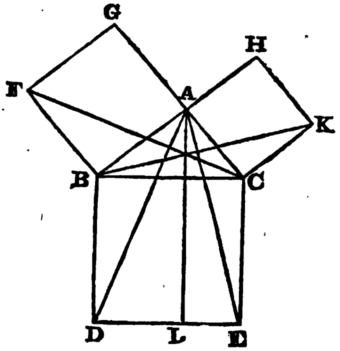
The demonstrations in the text of these two propositions are much simpler than those usually given; this is more particularly the case with proposition IX., which Euclid and succeeding authors demonstrate by means of the property of the square of the hypothenuse. In the demonstration here given this property is dispensed with, and the reasoning is, moreover, shortened, as will appear by comparing it with proposition IX. of Euclid, or with proposition XVIII. of Leslie.

On Proposition X.

This celebrated property, the discovery of which is attributed to Pythagoras, may be demonstrated in various ways: Euclid's demonstration, which is very elegant, is as follows:—

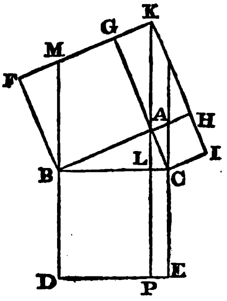
Let ABC be a right angled triangle having the triangle BAC; the square described upon the side BC is equivalent to the squares described upon BA, AC.

On BC describe the square BDEC, and on BA, AC the squares GB, HC; and through A draw AL parallel to BD or CE; join AD, FC: then, because each of the angles BAC, BAG is a right



angle, the two straight lines AC, AG upon the opposite sides of AB make with it, at the point A, the adjacent angles equal to two right angles; therefore CA is in the same straight line with AG; for the same reason, AB and AH are in the same straight line; and because the angle DBC is equal to the angle FBA, each of them being a right angle, add to each the angle ABC, and the whole angle DBA is equal to the whole FBC; and because the two sides AB, BD are equal to the two FB, BC, each to each, and the angle DBA equal to the angle FBC; therefore the base AD is equal to the base FC, and the triangle ABD to the triangle FBC. Now the parallelogram BL is double of the triangle ABD, because they are upon the same base BD, and between the same parallels BD, AL; and the square GB is double of the triangle FBC, because these also are upon the same base FB, and between the same parallels FB, GC. But the doubles of equals are equal to one another: therefore the parallelogram BL is equal to the square GB. And, in the same manner, by joining AE, BK, it is demonstrated that the parallelogram CL is equal to the square HC. Therefore the whole square BDEC is equal to the two squares GB, HC; and the square BDEC is described upon the straight line BC, and the squares GB, HC upon BA, AC. Wherefore the square upon the side BC is equal to the squares upon the sides BA, AC.

The following method of demonstrating the same proposition also deserves notice for its simplicity :—



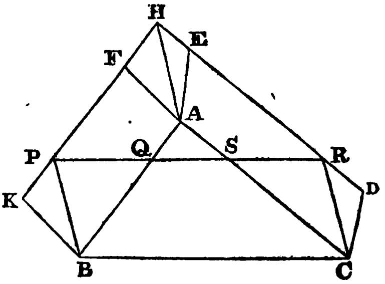
Let BE be the square on the hypothenuse, and BG, CH the squares on the sides. Produce DB to M, and through A draw PLA parallel to DB, and meeting the prolongation of FG in K.

Then, since the angles FBA, MBC are both right angles, if MBA be taken from each, there will remain the equal angles FBM, ABC; and, consequently, since the triangles FBM, ABC are both right angled, and have also the sides BF, BA equal, their hypothenuses BM, BC are equal (Prop. XI. B. I.); hence BM is equal to BD, in other words the rhomboids BK, BP, of which the common altitude is BL, have equal bases; these rhomboids, therefore, are equivalent. But the rhomboid BK is equivalent to the square AF, for they have the same base BA, and the same altitude BF. It follows, therefore, that the rhomboid BP is also equal to the square AF. In like manner, if IH be produced to meet the prolongation of PLA, it may be shown that the rectangle CP is equivalent to the square AI. Consequently the two rectangles BP, CP, that is, the square BE, is equivalent to both the squares AF, AI.

Since the converse of the proposition is true, this property belongs exclusively to the right angled triangle. But the following kindred property pertains to every triangle, viz.:—

In any triangle ABC, if rhomboids BF, CE be constructed on the sides BA, CA, and if KF, DE meet when prolonged in H, and CR parallel to HA be drawn, then the rhomboids BF, CE will, together, be equivalent to the rhomboid whose adjacent sides are BC, CR.

Draw BP parallel to CR, and join PR.



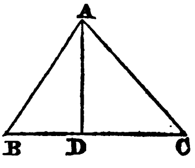
Then both CRHA and BPHA are rhomboids; consequently CR and BP are each equal to HA, and as they are also parallel, PBCR is a rhomboid.

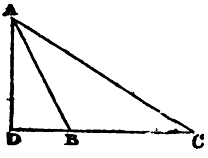
Now the rhomboids BKFA, BPHA are equal, as they have the same base BA, and are between the same parallels. For similar reasons the rhomboids CDEA, CRHA are equivalent; consequently the sum of the rhomboids BKFA, CDEA is equivalent to the space BPHRCA. Again, the triangles ABC, HPR are equiangular on account of the parallels, and as the bases BC, PR are equal, these triangles must be equal; hence, taking the portion AQS from each, there remains the quadrilateral BS equivalent to the space QPHRSA; and, consequently, by including the triangles PBQ, RCS, the rhomboid PBCR is equivalent to the space BPHRCA, or to the two rhomboids BF, CE.

If the line PR be situated above the vertex A of the triangle ABC, then instead of deducting a triangular space AQS, we shall have to add that included by the lines BP, PR, RC, and the sides CA, AB of the triangle.

From the last corollary annexed to proposition X. may be readily deduced a convenient method of determining the *area* of a triangle, from having the numerical values of the three sides given.

Thus, in the triangle ABC, the expression $\frac{AC^2 - AB^2}{2BC} + \frac{1}{2}BC$, denotes the value of the greater portion DC, into which the base is divided by the perpendicular AD; and consequently this perpendicular becomes at once determinable from the proposition itself: it is equal to $\sqrt{AC^2 - DC^2}$; having then the base and perpendicular of the triangle ABC, half their product gives the area.





The first of the above expressions results from the two simple numerical properties, that the difference of the squares of two numbers, divided by the sum of those numbers, gives their difference for the quotient; and again, that half the sum of two numbers added to half their difference, gives the greater of the two numbers. Now by the corollary, $AC^2 - AB^2$ is equivalent to $DC^2 - DB^2$; so that we have, in virtue of the first property just adverted to, $\frac{AC^2 - AB^2}{BC} = DC - DB$, and therefore from the second property $\frac{AC^2 - AB^2}{2BC} + \frac{1}{2}BC = DC$.

Suppose, for example, that $AB = 10$, $AC = 17$, and $BC = 9$, then $\frac{17^2 - 10^2}{18} + \frac{9}{2} = \frac{108}{18} = 6 = DB$, and $\sqrt{(10^2 - 6^2)} = \sqrt{64} = 8 = AD$; therefore $\frac{9 \times 8}{2} = 36$ the area of the triangle.

On Proposition XIII.

This property may be obtained in a different manner. Instead of deducing the property of the triangle which forms the first corollary from that of the rhomboid, the usual method is to infer the property of the rhomboid from that of the triangle: thus

In any triangle ABC, if a straight line AE be drawn from the vertex to the middle of the base, we shall have

$$AB^2 + AC^2 = 2AE^2 + 2BE^2$$

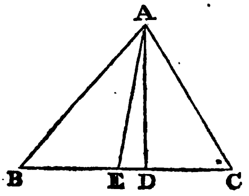
On BC let fall the perpendicular AD.

The triangle AEC (Prop. XI.) gives

$$AC^2 = AE^2 + EC^2 - 2EC \times ED.$$

The triangle ABC (Prop. XII.) gives

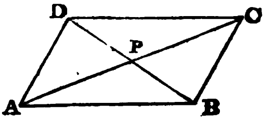
$$AB^2 = AE^2 + EB^2 + 2EB \times ED.$$



Hence, by adding the corresponding sides together, and observing that EB and EC are equal, we have

$$AB^2 + AC^2 = 2AE^2 + 2EB^2.$$

Cor. Hence, in every parallelogram, the squares of the sides are together equal to the squares of the diagonals.



For the diagonals AC, BD bisect each other; consequently the triangle ABC gives

$$AB^2 + BC^2 = 2AP^2 + 2BP^2.$$

The triangle ADC gives, in like manner

$$AD^2 + DC^2 = 2AP^2 + 2DP^2.$$

Adding the corresponding members together, and observing that BP and DP are equal, we shall have

$$AB^2 + AD^2 + DC^2 + BC^2 = 4AP^2 + 4DP^2.$$

But $4AP^2$ is the square of 2AP, or of AC; and $4DP^2$ is the square of BD; hence the squares of the sides are together equal to the squares of the diagonals.

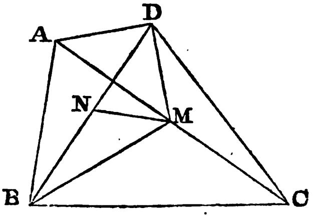
Upon comparing the two methods, that in the text appears to be somewhat simpler.

This property of the rhomboid results also from a more general property of the quadrilateral first demonstrated by *Euler,* by help of the foregoing proposition concerning the triangle.

In any quadrilateral the squares of the sides are equivalent to the squares of the diagonals, together with four times the square of the line joining their middle points.

Let ABCD be any quadrilateral, and M, N the middle points of the diagonals AC, BD, then the sum of the squares of the four sides of the quadrilateral will be equivalent to the squares of the diagonals, together with four times the square of the line MN.

Draw BM, DM: then, from the triangles ABC, DAC, we have, by the proposition above,



$$AB^2 + BC^2 = 2AM^2 + 2BM^2;$$

$$AD^2 + DC^2 = 2AM^2 + 2DM^2.$$

Consequently, by addition, there results

$$AB^2 + BC^2 + AD^2 + DC^2 = 4AM^2 + 2BM^2 + 2DM^2 = AC^2 + 2BM^2 + 2DM^2.$$

But from the triangle MBD there results

$BM^2 + DM^2 = 2BN^2 + 2MN^2$; hence $2BM^2 + 2DM^2 = BD^2 + 4MN^2$.

It therefore follows that

$$AB^2+BC^2+AD^2+DC^2=AC^2+BD^2+4MN^2.$$

When the points M, N coincide, then the quadrilateral becomes a rhomboid, and since, in that case, M N is nothing, the result exhibits the property before demonstrated.

It may not be amiss here to make known another very general and kindred property of the rhomboid, depending, like the preceding, upon the foregoing property of the triangle: I am unable to say whether or not it has been before noticed, I have seen it only in the particular case of the rectangle.

If from any point whatever lines be drawn to the four corners of a rhomboid, twice the sum of their squares will be equivalent to the squares of the diagonals, together with eight times the square of the line drawn from the given point to the intersection of the diagonals.

Let lines be drawn from the point P to the corners of the rhomboid ABCD, and to the intersection E of the diagonals.

Then from the triangle PDB we have

$$PD^2+PB^2=2DE^2+2PE^2;$$

and from the triangle PCA we have

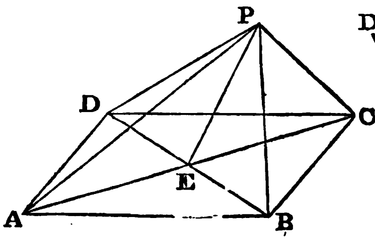
$$PC^2 + PA^2=2AE^2 + 2PE^2;$$

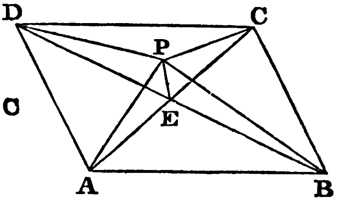
hence

$$PD^2 + PB^2 + PC^2 +PA^2=2DE^2+2AE^2+4PE^2;$$

consequently

$$2(PD^2 + PB^2 + PC^2 + PA^2)=DB^2 + AC^2 + 8PE^2$$





Cor. 1. If the rhomboid is a rectangle, then since DB = AC, it follows that the squares of the lines drawn from P to the corners are equivalent to the square of a diagonal, together with four times the square of the line drawn from P to its middle point.

Cor. 2. Also, since in the rectangle DE = AE, it follows, from the two first equations, that the squares of the lines drawn from P to two opposite corners, are equivalent to the squares of the lines drawn from the same point to the other two opposite corners.

It may be remarked, that if the point P be supposed to be situated at one of the corners of the rhomboid, we shall, as in the preceding general property of the quadrilateral, arrive at the relation already established between the squares of the sides and the squares of the diagonals.

From this proposition results also another curious property:—*If from the middle of a rhomboid, as a centre, a circle be described with any radius, the squares of lines drawn from any point in the circumference to the four corners of the rhomboid will always amount to the same sum.*

This may readily be converted into a *Porism* *, viz.,

Two points and the centre of a circle, not in the same straight line as the points, being given, two other points may be found, such, that if from the given centre with any radius, a circle be described; the squares of the lines drawn from the four points, to any point in the circumference, will amount to the same sum.

I shall conclude this note by mentioning one more property, readily derivable from the theorem respecting the triangle.

If two concentric circles be described, then from whatever point in the circumference of the one, lines be drawn to the extremity of any diameter of the other, the sum of their squares will always be the same.

BOOK III.

On Propositions XI. XII. and XIII.

In the scholium to the first of these propositions, the restriction in the enunciation was shown to be necessary. In Leslie's Geometry this restriction has been improperly omitted; a similar omission has been made also in Corollary 1, of the first Proposition, in the sixth book of the same work, where it is inferred that "straight lines which cut diverging lines proportionally are parallel," although it is obvious that these straight lines may cross each other. Such inadvertencies in elements of geometry are of consequence, however trifling they may appear to some, and certainly stand in need of correction.

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* A Porism is "a proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate, or capable of innumerable solutions." This definition of a porism is given by Professor Playfair in vol. 3. of the Transactions of the Royal Society of Edinburgh, in his paper "On the origin and investigation of Porisms;" to which the student may refer for valuable and important information upon a highly curious and interesting subject.

In the twelfth and thirteenth propositions, and their corollaries, are comprehended some useful truths relative to the contact and intersections of circles, upon which several of the problems in the fourth book depend.

These two propositions and corollaries include the 11th, 12th, 13th, and 14th propositions of Legendre's second book, and also the converse of those propositions.

On Propositions XIV. XV. XVI. and XVII.

The way in which Euclid has enunciated and demonstrated the first of these propositions, has rendered it necessary that he should, in the succeeding proposition, demonstrate that "The angles in the same segment of a circle are equal to one another," a proposition which presented two cases requiring separate consideration. It was absolutely necessary that this truth should be established; but it could not be inferred from the preceding proposition, without considering *re-entrant* angles, which Euclid has made no mention of throughout his Elements. The demonstration given in the text equally avoids the introduction of re-entrant angles, while it dispenses with the proposition which Euclid found it necessary afterwards to demonstrate, as this immediately follows as an obvious inference from the proposition itself.

The fifteenth proposition may appear, perhaps, rather less simple than the corresponding proposition in Euclid, as an additional line has been introduced into the diagram. But I deemed it preferable to establish this proposition before proposition XVII. instead of adopting Euclid's course, and this preference has been given on the ground of simplicity; for this seventeenth proposition, by aid of the foregoing property, becomes susceptible of a much easier demonstration than that which Euclid gives, as will appear from a comparison of Euclid's demonstration with that in these elements. With regard to the sixteenth proposition, it seems necessary merely to remark, that I have endeavoured to combine, in a single train of reasoning, all the various cases that the proposition presents, and I am in hopes that this reasoning will be found conclusive.

BOOK IV.

The propositions in this book are all problems, in which every practical operation that in the course of the preceding books was admitted to be possible, is actually performed. In Euclid's Elements, the problems are interspersed among the theorems, in order that in every demonstration no operation may be supposed possible, that has not been previously effected, with a view no doubt, as Mr. Playfair observes, "to guard against the introduction of impossible hypotheses, or the taking for granted that a thing may exist, which, in fact, implies a contradiction." There are some advantages, however, connected with a different arrangement; for, by thus keeping the theorems and problems distinct, a continuity is preserved in the chain of reasoning, and the mind proceeds from one truth to another without being interrupted by any thing of a mechanical nature; and, moreover, the problems themselves become, by this separation, susceptible of simpler and easier constructions, because we are enabled to avail ourselves of a greater number of previously established principles.

The restriction which Euclid has put upon himself, in this respect, appears to be unnecessary. The learned writer just quoted remarks, that "this rule is not essential to geometrical demonstration, where, for the purpose of discovering the properties of figures, we are certainly at liberty to suppose any figure to be constructed, or any line drawn, the existence of which does not involve an impossibility."

In the construction of problems, however, the case is widely different; for we cannot admit any preliminary construction till it has been actually effected. Every geometrical problem must remain unsolved, while it involves in it the trisection of an angle, since this operation cannot be actually performed; but if the supposition of such trisection, in the course of any demonstration, were necessary to the establishment of a theorem, the conclusion would be as true and as satisfactory as if the above problem presented no difficulty; all this must be quite obvious, for the truths of geometry necessarily exist independently of any practical operations, and may, therefore, be reached without their aid.

The twentieth problem in this book was taken from the Ladies' Diary, where it was proposed and solved by Mr. Dotchen: the construction here given is somewhat simpler than his.

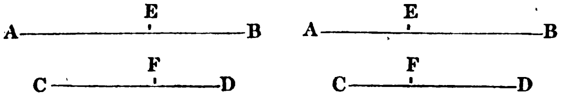
BOOK V.

This book is occupied by the doctrine of proportion, a subject of the highest importance, not only in geometry, but in every part of the mathematics. The importance of the subject is not, however, greater than the difficulties that have hitherto attended it; difficulties, the removal of which has resisted the attempts of geometers for a period of more than two thousand years—from the time of Euclid down to the present. Euclid establishes the doctrine of proportion in his fifth book, by reasonings of the most rigorous character, and in a manner so general and comprehensive, that magnitudes of all kinds are included without any restrictions or arbitrary conditions whatever. These reasonings, however, are so exceedingly subtle, and it must be confessed, in some instances, so obscure, arising from the metaphysical considerations which they involve, that many, having been unable fully to enter into the spirit of it, have mistrusted his conclusions, and have ventured rashly to question their legitimacy. These circumstances have naturally drawn the attention of succeeding geometers to the formation of a treatise of proportion, of the same extent and universality as that of Euclid, in which the intricacies of his method might be avoided. But all attempts to accomplish this object have either entirely failed, or only very partially succeeded; so that, at the present day, there exists no rigorous and universal treatise on geometrical proportion except the fifth book of Euclid's Elements.

Many and important have been the errors into which geometers have fallen in their deviations from Euclid, on the subject of proportion. This assertion will be supported by a reference to some of the most reputable productions of the present day, where Euclid's conclusions have been reached by a shorter path, but by unwarrantable steps in the reasoning, which have consequently rendered those conclusions, though true, illegitimate.

Take, for instance, proposition XVI. of the fifth book of Bonnycastle's Geometry, viz.:—

If four magnitudes be proportional, the sum of the first and second will be to the first or second as the sum of the third and fourth is to the third or fourth.



Let AE be to EB as CF is to FD; then will AB be to BE, or AE, as CD is to DF, or CF.

For since AE is to EB as CF is to FD; therefore, alternately, AE will be to CF as EB to FD. And, since the antecedent is to its consequent as all the antecedents are to all the consequents, AE will be to CF as AB is to CD.

But ratios, which are the same to the same ratio, are the same to each other; whence AB will be to CD as EB is to FD; and, alternately, AB to EB as CD to DF.

Again, since AE has been shown to be to CF as AB is to CD; therefore, by alternation, AE will be to AB as CF is to CD. But quantities, which are directly proportional, are also proportional when taken inversely; whence AB will be to AE as CD is to CF.

Now this conclusion is not legitimate, for the above reasoning, if meant to be general, is altogether inadmissible; as will appear obvious from observing, that in every step except the last, the alternation of the proportionals is required, which alternation is not possible except when the magnitudes are all of the same kind; this demonstration, therefore, applies only to a particular case of the proposed theorem. The next proposition in the same work, viz., proposition XVII. is, in like manner, conclusive only in the particular case when the magnitudes are all of the same kind. Mr. Bonnycastle, however, so far from being aware of this circumstance, attributes to these conclusions the same generality that belongs to Euclid's; nay, indeed, he asserts in his notes that they are even more general than those of the Greek geometer; for he says, "It has been properly observed by Mr. Simpson that the manner in which the composition and division of ratios is treated of by Euclid is defective, as not being sufficiently general. It is also commonly found very abstruse and embarrassing to beginners, on account of the complicated terms in which it is enunciated, and the number of cases to be separately demonstrated. For these reasons it was deemed necessary to give the propositions a more simple and *general* form, and to render the demonstrations of them as concise and perspicuous as possible."

Having copied this note, it is incumbent on me to add, that Euclid's method (as restored by Dr. Simson) of treating the composition and division of ratios, so far from being defective and not sufficiently general, is undoubtedly complete and universal; and in justice to Mr. Thomas Simpson, as well as to Euclid, I must observe that the remark which Mr. Bonnycastle attributes to the former was in reality never made. It is remarkable that the two propositions just noticed should have been allowed to pass as genuine for so long a period (about 30 years), in a book of such popularity as Bonnycastle's Geometry; and that the foregoing unjust remarks upon the accurate reasonings of Euclid, with which the name of Mr. Simpson is so unwarrantably coupled, should have hitherto escaped the censure which they deserve.

As another example of this inconclusive reasoning, we may refer to Professor Leslie's manner of treating this subject in his Elements of Geometry, where the propositions on proportion are demonstrated to be true only when the magnitudes are both *commensurable* and *homogeneous;* that these demonstrations do not extend to incommensurable magnitudes the learned professor seems well aware, but it does not appear that he is also aware of their being restricted to homogeneous magnitudes. That such is the case, however, may be readily shown:—Take, for example, proposition XV. of his fifth book.

If two analogies have the same antecedents, another analogy may be formed having the consequents of the one for its antecedents, and the consequents of the other for its consequents.

Let A : B :: C : D, and A : E :: C : F; then B : E :: D : F.

For alternating the first analogy, A : C :: B : D*, and alter-

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[*] According to Mr. Leslie's own definition of proportion, this proportion will be mpossible, unless the terms of the proportion A : B :: C : D are homogeneous.

nating the second, A : C :: E : F; whence, by identity of ratios, B : D :: E : F.

This reasoning is conclusive only in the particular case mentioned above, as it is liable to the same objections that have already been shown to belong to Bonnycastle's reasoning. The modern French writers on proportion all treat the subject in the same objectionable manner; their reasonings being applicable only to magnitudes which are at once commensurable and homogeneous, or to mere numbers which are necessarily thus restricted. Legendre, however, must be excepted from this remark, for in his geometry he has not treated on the doctrine of proportion at all! but has referred for information on this subject to sources which cannot possibly supply it, viz.: "to the common treatises on arithmetic and algebra," which, as every one knows, relate only to number. *Dr. Brewster* has observed, in the introduction to his translation of Legendre, that "the author has provided for the application of proportion to incommensurable quantities, and demonstrated every case of this kind as it occurred, by means of the *reductio ad absurdum*." This assertion, however, I must venture to dispute:—How, for instance, is the truth of the corollary to proposition XXVIII. of his third book shown when the lines concerned are incommensurable? or the corollary to the next proposition in like circumstances? It will be found upon examination that in these, and in many other cases, the inferences hold good only when the magnitudes are commensurable. Legendre's geometry is, therefore, in this respect very defective; and it is to be regretted that this able geometer did not apply his powerful talents to a subject of so much difficulty and importance.

In the treatise on proportion which I have given, I have endeavoured to treat the subject in the same rigorous and comprehensive manner as Euclid, and that without employing the *reductio ad absurdum* so often, even as Euclid himself employs it in his fifth book. Propositions IX., X., and XVIII. of Euclid are demonstrated by means of this principle*; in these elements only two propositions involve this principle, viz., propositions III. and IV. I have too, I hope, succeeded in some measure in my attempts to strip the subject of much of its difficulty; as I have been enabled to reach Euclid's conclusions without the aid of his subsidiary propositions relative to

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* Dr Barrow's Euclid, or the French translation of *M. Peyrard*.

ratios and their comparison, propositions in which his greatest subtilties of reasoning are involved. I have wholly abandoned the use of this term *ratio* in these elements; the term in reality denotes the quotient arising from the division of one magnitude or quantity by another of the same kind; it is accurately assignable when the magnitudes are commensurable, but unassignable when they are incommensurable. Euclid's definition of *ratio* is obscure and unnecessary, and his doctrine of ratios has given rise to long commentaries and discussions, which have, perhaps, rather increased than diminished the perplexity of the subject. That there is no absolute necessity for the consideration of ratios in elements of geometry, I have, I hope, satisfactorily shown in the present treatise.

I am bound to acknowledge that this method of treating proportion was at first suggested to me by an examination of a brief tract on the same subject, contained in the *Principes Mathématiques* of M. da Cunha, a work of great ingenuity, and which was first introduced to the notice of the English student by Professor Playfair, in vol. xx. of the Edinburgh Review. This treatise on proportion, although but an epitome of the subject, was highly commended by Mr. Playfair, especially the neat definition which M. da Cunha had given of proportional magnitudes. This definition, however, Mr. Playfair emended, and in this improved form I have adopted it.

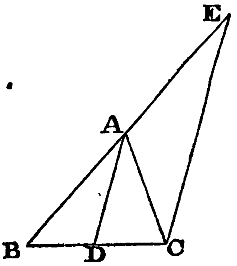
The reasoning of M. da Cunha, although free from the embarrassments attendant upon the comparison of ratios, is, nevertheless, in certain propositions exceedingly intricate; propositions VI. and VII., for instance, which correspond to the IVth and VIth in these elements, are incomparably more perplexing than these latter. On this account I found myself compelled to relinquish the intention I had originally formed of following this author's steps, and resolved to proceed in an entirely different manner; the result is the treatise in the text, which is not only more simple, but much more comprehensive than that of M. da Cunha, and which I have taken considerable pains to render deserving of the approbation of geometers.

On Proposition XIX.

This proposition is, I believe, new; and by means of it the demonstration of proposition XXXI. of the sixth book is rendered much more simple and concise than any that has previously been given.

BOOK VI.

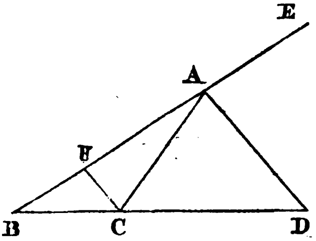
On Proposition VII.



This proposition affords a simple method of bisecting an angle:—Thus, if the angle BAC is to be bisected, it will be only necessary to produce one side, as BA, till AE be equal to AC, and then to draw AD parallel to the line joining EC: an operation very expeditiously performed by means of a *parallel ruler*.

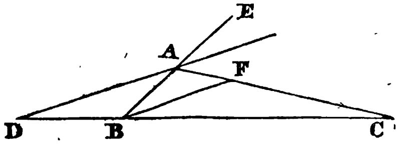
The proposition in the text may be extended to the cases where the exterior angle CAE is bisected; for it may be proved that if the bisecting line cuts the extension of the base, the parts intercepted between it and the sides are also as those sides. This proposition presents two cases which *Dr. Simson* seems not to have observed, as he has demonstrated one case only *.

Let AD bisect the exterior angle CAE, and let it cut the base produced in the point D, then BD : DC : : BA : AC.



For, draw CF parallel to DA; then (Prop. V. Cor. B. VI.) BD : DC : : BA : AF; also, because CF, DA are parallel, the angles CFA, DAE are equal; so, also, are the angles FCA, DAC, but, by hypothesis, the angles DAE, DAC are equal; hence the angles CFA, FCA are equal, and, therefore, AC = AF; hence, putting AC for AF in the above proportion,

BD : DC : : BA : AC.



If, now, AD cut the base produced on the other side of AC, which is the second case of the theorem; then, instead of drawing a parallel to it from the point C, it must be drawn from B, as in the annexed diagram; after which the demonstration will be similar to that of the foregoing case. The converse is true, and may be easily proved.

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* Simson's Euclid, proposition A, book VI.

If the triangle ABC be isosceles, the line bisecting the exterior angle CAE will obviously be parallel to the base.

On Proposition XII.

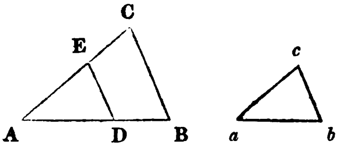
This proposition is the same as the seventh of Euclid's sixth book, but is here demonstrated in a manner far more simple and concise. Mr. Thomas Simpson in his Elements of Geometry has rejected this proposition of Euclid, and has in its place substituted another which is absolutely false! It is a singular circumstance that this important error should have hitherto escaped detection, appearing as it does in a work of such high repute as Simpson's Geometry, which has been in the hands of every mathematician in Europe for the greater part of a century. One can scarcely imagine indeed how it could have passed the scrutiny of Dr. Robert Simson, who, it is well known, indulged no very friendly feeling towards his cotemporary, whom he viewed as an opponent because he had ventured to find fault with one or two of Euclid's demonstrations. The same erroneous proposition appears also in Mr. Leslie's Geometry, Prop. XIV. B. VI., last edition: in the preceding edition it is proposition XV.; and although the reasoning is different from Mr. Simpson's, it is equally inadmissible. It will be necessary to show this.

THEOREM XVI. Book IV. Simpson's Geometry.

If two triangles (ABC, *abc*) have one angle (A) in the one equal to one angle (*a*) in the other, and the sides (AB, *ab*, CB, *cb*) about either of the other angles proportional; then will the triangles be equiangular, provided these last angles (B, *b*) be either both less, or both greater than right angles.

In AB, let AD be taken $= ab$, and let DE be drawn parallel to BC, meeting AC in E.

Then will the triangles ABC and ADE be equiangular; therefore $CB : ED :: AB : AD :: AB : ab :: CB : cb$, and, consequently $ED = cb$; *whence the triangles abc and ADE*

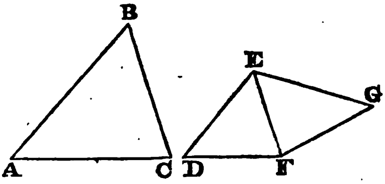


(*having* $ab = AD$, $cb = ED$, *and* $a = A$) *will be equal in all respects, provided the angles* abc *and* ABC ($= ADE$) *are either both less, or both greater than right angles.* Therefore, since the latter of these equal triangles (abc, ADE) is equiangular to ABC, the proposition is manifest.

PROPOSITION XIV. THEOREM. Leslie's Geometry, Book VI.

Triangles are similar which have each an equal angle, and the sides containing another angle of the same character proportional.

Let the triangles CAB and FDE have the angle ABC equal to DEF, and the sides that contain the angles at C and F proportional, or BC : AC :: EF : FD; while those angles are both of them either acute or obtuse, the triangles ABC and DEF are similar.



For, from the points E and F, draw EG and FG, making the angles FEG and EFG equal to ABC and BCA.

The triangle ABC is evidently similar to GEF, and BC : CA :: EF : FG; but, by hypothesis, BC : CA :: EF : FD; and, therefore, EF : FG :: EF, FD, and FG is equal to FD. *Whence the triangles EGF and EDF, having the angle FEG equal to FED, the side FG equal to FD, and the side EF common, and being both of the same character with CAB, are equal* * ; consequently the angle GFE or ACB is equal to DFE; and, therefore, the triangles ABC and DEF are similar.

In these reasonings the steps printed in italics are erroneous: proposition XXVI. of the first book, on which they are made to depend, does not warrant the conclusions to which they lead. These geometers might have observed that if this proposition were true, it would have followed from proposition VII. of this book, that the line which bisects any angle of an acute-angled triangle, or the obtuse angle of an obtuse-angled triangle, divides the triangle into two similar triangles, which, however, is only true of the isosceles triangle. It seems scarcely necessary, formally, to prove the falsity of the proposition in question; I shall, however, for the sake of the young student, give the following simple illustration:—

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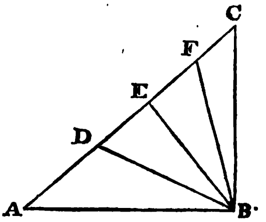
* The reference made here is to proposition XXVI. book I.



Let the triangle ABC be right-angled at B; draw BE perpendicular to AC: take the points F, D equidistant from E, and join B, F; B, D; then the triangles FCB, DCB have an angle C in each equal, and the sides containing the acute angles FBC, DBC proportional, and yet they are not similar; for the one triangle has an obtuse angle and the other has not.

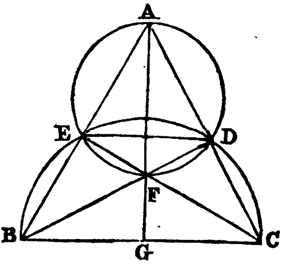
The various scholia interspersed through this sixth book render any further observations here unnecessary. It is hoped that among those scholia will be found some new and important remarks, calculated to give the student a more accurate and comprehensive view of the application of proportion, than elements of geometry usually furnish.

To this book a valuable proposition or two may be added.

PROPOSITION I.

If from the vertices of the three angles of a triangle perpendiculars be drawn to the opposite sides, they will intersect in the same point, and the rectangle of the parts shall be the same in each.

In the triangle ABC, let the perpendiculars BD, CE intersect each other in F; draw AF, and produce it, if necessary, till it meets BC in G: AG will be perpendicular to BC.



For draw DE: then since the opposite angles AEF, ADF, of the quadrilateral AEFD, are equal to two right angles, a circle AEFD may be circumscribed about it. Also, since the angle BEC is right, BC will be the diameter of the circle which circumscribes the triangle EBC; for a similar reason it will also be the diameter of the circle which circumscribes the triangle DCB; hence the circle BEDC circumscribes both triangles. Now the angles BCE, BDE, subtended by the same arc BE, are equal; but the angles FDE, FAE, subtended by the arc FE, are also equal; hence the angle BCE is equal to the angle BAG; and since the triangles BCE, BAG have also the common angle B, it follows that the third angles BEC, BGA are also equal; hence AG is perpendicular to BC.

Again, by Prop. XXIV. B. VI., $BF \cdot FD = CF \cdot FE$; and since the triangles CFG, AFE are similar, CF : FG :: AF : FE; therefore $CF \cdot FE = AF \cdot FG$; consequently,

$$BF \cdot FD = CF \cdot FE = AF \cdot FG.$$

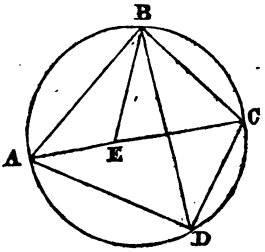
Cor. By Prop. XXVI. Cor. 2. B. VI., $AB \cdot AE = AC \cdot AD$, and therefore AB : AC :: AD : AE; hence (Prop. XI. B. VI.) the triangles ABC, ADE are similar.

There is another case of this theorem, viz., that in which the point F is without the triangle. As the reasoning will in this case be very similar to the preceding, I shall leave it for the student to supply, merely hinting that, in the last step, the reference will be to Prop. XXVI. Cor. 2., and not to Prop. XXIV. as above.

PROPOSITION II.

The rectangles of the opposite sides of a quadrilateral inscribed in a circle, are together equivalent to the rectangle of the diagonals.

Let ABCD be an inscribed quadrilateral; draw the diagonals AC, BD, then $AB \cdot CD + AD \cdot BC = AC \cdot BD$.



For draw BE, so that the angle ABE may be equal to the angle CBD, then the angle ABD is equal to the angle CBE, and the angles BDA, BCE being in the same segment, are also equal; therefore the triangles ABD, BCE are similar; hence AD : BD :: EC : BC, and consequently, $AD \cdot BC = BD \cdot EC$.

Again, since the angles ABE, DBC are equal, as also the angles BAE, BDC, being in the same segment, the triangles AEB, DCB are similar, so that AB : AE :: BD : CD; therefore $AB \cdot CD = AE \cdot BD$. Consequently, $AB \cdot CD + AD \cdot BC = AE \cdot BD + EC \cdot BD = AC \cdot BD$.

Schol. From this property may very readily be derived several important trigonometrical formulæ.

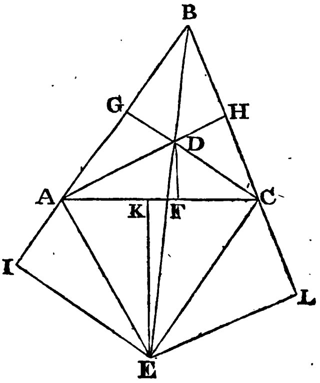
PROPOSITION III. PROBLEM.

It is required to determine the numerical expression for the surface of a triangle, from having the three sides given.

Let the three sides of the triangle BAC be given.

Produce the sides BA and BC, draw the straight lines BE, AD, and AE, bisecting the angles CBA, BAC, and CAI; join CD and CE, and let fall the perpendiculars DF, DG, and DH within the triangle, and the perpendiculars EI, EK, and EL without it.

The triangles ADF and ADG, having the angle DAF equal to DAG, the angles F and G right angles, and the common side AD, are equal; for the same reason the triangles BDG and BDH are equal. In like manner it is proved that the triangles AEL and AEK are equal, and the triangles BEI and BEL. Whence, the triangles CDH and CDF, having the side DH equal to DF, the side DC common, and the right angle CHD equal to CFD,—are equal; and for the same reason, the triangles CEK and CEL are equal. The perimeter of the triangle ABC is therefore equal to twice the segments AF, FC, and BG; consequently BG is the excess of the semi-perimeter above the base AC, and AG is the excess of that semi-perimeter, or of the segments BH, HC, and AG, above the side BC. But the sides AB and BC, with the segments AK, and CK, or AI and CL, also form the perimeter; whence BI being equal to BL, the part AI is the excess of the semi-perimeter above the side AB.



Now because DG and EI, being perpendicular to BI, are parallel, BG : DG :: BI : EI, and consequently BI × BG : BI × DG :: DG × BI : DG × EI. But since AD and AE bisect the angle BAC and its adjacent angle CAI, the angles GAD and EAI are together equal to a right angle, and equal therefore to IEA and EAI; whence the angle GAD is equal to IEA, and the right-angled triangles DGA and AIE are similar; wherefore DG : AG :: AI : EI, and DG × EI = AG × AI; consequently BI × BG : DG × BI :: DG × BI : AG × AI. But the triangle ABC is composed of three triangles ADB, BDC, and CDA, which have the same altitude; and therefore its area is equal to the rectangle under DG, and half their bases, AB, BC, and AC, or the semi-perimeter BI. Whence the area of the triangle ABC is a mean proportional between the rectangle under BI, and its excess above AC, and the rectangle under its excess, above BC, and that above AB.

It appears, therefore, that the area of a triangle will be expressed numerically, by the square root of the continued product of the semi-perimeter, into its excesses, above the three sides. A commodious practical rule is thus furnished for finding the area of a triangle, from having its three sides given. There are several ways of obtaining this rule, but the above investigation, which I have extracted from Mr. Leslie's Geometry, may be reckoned among the simplest.

The following expression, in which p represents the semi-perimeter; a, b, and c, the three sides, and A the area of the triangle, exhibits this rule in a form more easily retained in the memory:

$$A = \sqrt{p\,(p-a)\,(p-b)\,(p-c)}.$$

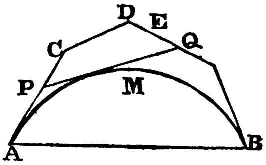
BOOK VII.

In this book, as well as in the preceding, will be found some remarks that appear well worthy of the student's attention. These consist chiefly of an examination of the circumstances in which certain properties become less general, or fail altogether; and they will, I think, be often found to unfold curious and interesting particulars, all more or less tending to enlarge the student's views of the subject.

In the scholium to proposition IX., an attempt is made to deduce, by a simple train of reasoning, the well known axiom of Archimedes, viz., that the circumference of a circle is a shorter line than the perimeter of any circumscribed polygon, and a longer line than the perimeter of any inscribed polygon. The arguments from which I have deduced this conclusion, cannot, I must confess, claim the character of rigorous demonstration; the proposition indeed, is such, as not to admit of a rigid proof, but I think the method which I have adopted affords as much evidence of its truth as can be given. *M. Legendre*, in proposition IX. of his fourth book, has attempted to demonstrate this proposition, but his reasoning appears to be inconclusive. It is as follows :

Any curve, or any polygonal line which envelops the convex line AMB from one end to the other, is longer than AMB, the enveloped line.

We have already said, that by the term convex line, we understand a line, polygonal, or curve, or partly curve and partly polygonal, such that a straight line cannot cut it in more than two points. If in the line AMB there were any sinuosities or re-entrant portions, it would cease to be convex, because a straight line might evidently cut it in more than two points. The arcs of a circle are essentially convex; but the present proposition extends to any line which fulfils the required conditions.

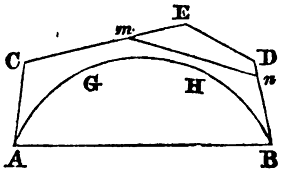


This being premised, if the line AMB is not shorter than any of those which envelop it, there will be found among the latter a line, shorter than all the rest, which is shorter than AMB, or at most, equal to it. Let ACDEB be this enveloping line: any where between these two lines draw the straight line PQ, not meeting, or at least only touching, the line AMB. The straight line PQ is shorter than PCDEQ; hence, if instead of the part PCDEQ, we substitute the straight line PQ, the enveloping line APQB will be shorter than APDQB. But, by hypothesis, this latter was shorter than any other; hence, that hypothesis was false; hence all the enveloping lines are longer than AMB.

Now all that this reasoning proves is, that it is impossible to find, among the enveloping lines, a line shorter than all the rest; for whatever line be supposed the shortest, one shorter still may always be found. If indeed such a line could be found, then, if the hypothesis that this line is shorter than, or equal to, AMB, could be shown to be impossible, the truth of the theorem would be indisputably established. Legendre has doubtless deceived himself in the foregoing reasoning; he has unconsciously set out with *two* hypotheses, and having shown *one* to be impossible, infers, unwarrantably, the impossibility of the other.

A more satisfactory method of obtaining this conclusion is given by

M. Develey, in the notes to his *Elémens de Géométrie*, the substance of which I shall here give.



Take either of the lines that can envelop the convex line AGHB, the line ACEDB for example; there will necessarily be some space between these two lines, otherwise they would be identical, and would, in reality, form but one. Through a point in this space draw a straight line, which may meet the enveloping line in two points *m*, *n*, but not cut the enveloped line. Then we shall have *mn* ∠ *m*ED*n*, and consequently AC*mn*B ∠ AC*m*E*n*B. This proves that whatever enveloping line is taken, a shorter can always be found.

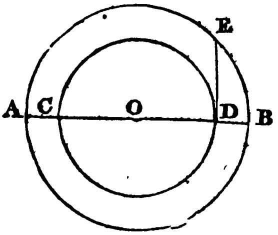
By repeating this construction, and proceeding from one enveloping line to a second and shorter line, from this to a third still shorter, and so on, we shall observe that the spaces inclosed by the enveloping lines become evidently smaller and smaller, and consequently always approach to the space contained by the enveloped line; that is, to the space AGHBA, which is doubtless smaller than either of the former. All these effects continue as long as there is any space between the enveloping line and the other. It appears, therefore, that since these enveloping lines decrease in length, as the space which separates them from the enveloped line diminishes, if there could be a last enveloping line it would in reality coincide with that enveloped; this therefore is shorter than all the rest.

BOOK VIII.

This book, like the fourth, is entirely practical: the construction of the fifth problem is taken from Mr. Leslie's Geometry. The last fifteen problems relate to the division of surfaces, an important part of practical geometry, although seldom noticed in the elements. The geometrical constructions of these problems I have framed so as to suggest the most convenient analytical solutions, in order that they may the more readily be applied in matters of real practice, such as the division of fields, &c. It does not appear necessary to exhibit here the analytical expression deducible from each construction, as they may be very easily inferred by the student.

On Propositions XXXIII. and XXXIV.

The area of the space included between two concentric circles, or of a circular ring, as it is sometimes called, is readily determined from knowing the diameters of the concentric circles.



Thus, if AB, CD be the respective diameters of two concentric circles, of which O is the common centre, then calling the surface of the inner circle s and that of the outer S, while π is put for the number 3·1416, we have, by the scholium to proposition XIII. of Book VII., the following expression for the surface of the ring, viz.,

$$S - s = \pi\,(OB^2 - OD^2)$$

Now

$$OB^2 - OD^2 = (OB + OD)\cdot(OB - OD) = AD\cdot DB;$$

also AD·DB is equal to the square of the perpendicular DE, which is a tangent to the inner circle at D, consequently

$$S - s = \pi\cdot DE^2;$$

that is, the surface of the ring is equivalent to that of a circle, whose radius is equal to the tangent DE.

The two problems in the text were first solved by the late venerable and ingenious Dr. Hutton, who appears to have set high value upon these solutions: the first solution was pirated very shortly after its publication by a Mr. Clark, whose conduct in this affair Dr. Hutton very indignantly reprobates, in the third volume of his valuable Mathematical Tracts; where an account of the origin of these problems is given at length, particularly of the last, which is by far the most curious and difficult of the two. I cannot resist the desire which I feel to gratify the student, by giving Dr. Hutton's account of this problem in his own interesting manner, with which extract I shall terminate these notes. After discussing the former problem, he proceeds thus:

"With respect to the other curious and kindred problem, that of dividing a given circle into any number of parts that may be mutually equal both in area and perimeter, some account of its rise has been already given. It was first anonymously proposed in the year 1774 as a curious paradoxical problem, but unaccompanied by the least hint or intimation of any mode of solution whatever. It was indeed, announced by the proposer expressly as a seeming paradox, but accompanied by the declaration, that it nevertheless was capable of a strict geometrical solution. The problem remained however some time unanswered, being given up by all persons as a matter quite hopeless; and by most deemed, in fact, as little to be expected as the quadrature of the circle itself, to which it was thought to be nearly allied, and indeed dependent on it; for no person could imagine any other possible way of a circle being divided, even in idea, into any number of such parts that might be equal, both in area and perimeter, than by radii drawn from the centre to the point of equal divisions in the circumference: this was, in effect, reducing the problem to this other, of dividing the circumference in *any* proposed number of equal parts, which was deemed on all hands a thing impossible to be effected. After some time no person thought any more of the matter, but as a thing never to be accomplished; and so I believe it might have remained to this day, but for the occurrence of some such accident as that which actually led myself into the train of thought which soon ended in the complete solution. The construction I first inserted in the Critical Review; next it was introduced into my first, or quarto volume of Tracts, published in the year 1786, accompanied with a short account of its rise, and a considerable improvement of it, by rendering the property general for the division into all ratios of parts, equal or unequal, and extending the same to all ellipses, as well as circles. After which, I have usually been in the habit of introducing it into my Dictionary, and the more common elementary books on mensuration," &c.

END OF THE NOTES ON THE FIRST EIGHT BOOKS.

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